Questions Pupils Ask!

"Why can't it be distance *plus* time?"

by **Colin Foster**

Suppose a pupil writes (or calculates) something like:

speed = distance + time.

Ouch! What is going on here? On a grainy photocopy of a worksheet, a ÷ sign may look a bit like a + sign, so perhaps that was the source of the mix-up? It can be tempting to seize on something like this as though it is a completely ridiculous statement. "How can you *add* a distance and a time?" the teacher might say. "They're in totally different units! What units would the answer be in?" It seems obvious that you can add things only when they have the same units, otherwise you're doing something that is dimensionally absurd. If only pupils would use their common sense and think about what they are doing. Can you add apples and oranges? Of course not!

But is this all really so obvious? After all, I *can* add apples and oranges to my shopping basket. Well, perhaps that's not really 'adding' in the mathematical sense, although adding is often treated as 'putting together'. I can certainly add *fruit* together, such as when I'm trying to reach my five portions of fruit and vegetables a day (Note 1). So, maybe adding apples and oranges is OK if you're treating 'fruit' as the category, because then you can add fruit to fruit? Or maybe, instead, we should say that we're adding the dimensionless *number of portions*, rather than adding actual 'fruit'?

School algebra really doesn't have much to do with apples and oranges. 'Fruit salad algebra', where something like $3a + 2b$ is taken to mean 'three apples and two bananas', is often invoked when collecting like terms ('If we have $2a + a$, then how many apples have we got altogether?'). However, this risks building up a 'letter-as-object' misconception (Küchemann, 1978), where pupils lose sight of the fact that a and b represent *numbers*. The shift from saying "So how many a's have we got?" to saying "So how many *lots* of a have we got?" can be valuable, because in the latter case a is more clearly a number, rather than an object. We might even have 'negative 3 lots of a'. This confusion about what letters stand for leads to people writing things like 'a chair has four legs' as $c = 4l$, only to find that, when they substitute in $l = 8$ legs, instead of getting 2 chairs, they get $c = 4 \times 8 = 32$ chairs! Trying to persuade pupils that $l = 4c$ is the right formula can be very difficult: "A leg equals four chairs? That's ridiculous!"

Treating letters as objects totally breaks down when you start to want to write things like ab or b^2 (a square banana?). Surely these things are 'obviously wrong' in the same sense that 'speed = distance + time' is 'obviously wrong'? And yet, in algebra, we *can* write things like

$a + ab = a(1 + b)$

if we want to, without worrying at all about dimensions. We often think of a letter as a length, and that seems to be OK (a length is not an 'object'?), but then when we write quadratic expressions like $x^2 + 2x + 1$, should we worry that we seem to be adding 'an area' to 'a length' to 'a dimensionless number'?

Pupils are often reluctant to define their letters when doing algebra. If they are doing routine exercises, the letters needed may already be defined in the question anyway, but in more substantial problems where pupils have to choose their own letters it is very important to be precise about what they mean. I have often found that if you insist that pupils define their letters they think that you are making a fuss about nothing, and so just write

'a = apples, b = bananas'

to please you, which actually doesn't help very much. Consider these two alternatives:

1. If a is the *price* of an apple and b is the *price* of a banana, then $3a + 2b$ is the cost of buying 3 apples and 2 bananas;

   ... whereas ...

2. If a is the *number* of apples and b is the *number* of bananas, then $3a + 2b$ pence is the cost of buying a apples and b bananas if they are being sold at the bargain price of 3 pence for an apple and 2 pence for a banana.

In both cases, the expression represents the cost, but on the one hand it is the cost of 3 apples and 2 bananas and on the other hand it is the cost of a apples and b bananas. So, writing 'a = apples, b = bananas' doesn't tell us which of these alternatives is the one we mean. These are quite different uses of *a* and *b* that arise because of the commutativity of something like 3*a*. You can perhaps get away with vague statements like '*a* = apples' when *a* refers to some quality of an apple (e.g. the cost or the weight), but not if *a* stands for the *number* of apples.

In the first case it would make no sense to write *ab* (how could you attach a meaning to multiplying the price of an apple by the price of a banana? – and the units would be 'square pence'!), but in the second case there is no logical reason why you couldn't write *ab*. For example, if you had *a* apples, each of a different variety, and *b* bananas, each of a different variety, then *ab* would the number of different-flavoured apple–banana smoothies you could make using one apple and one banana. I admit that that's a bit contrived, but the point is that it is not mathematically incorrect – and it certainly isn't absurd. In both cases, there is no problem with writing $a + b$: in the first case, it is the total cost of one apple and one banana, and in the second case it is just the total number of fruit.

So, the fact that the expression $a + b$ cannot be simplified doesn't really seem to have very much to do with saying that "you can't add apples and bananas". If *a* and *b* just represent some numbers, and we know what those numbers are, then $a + b$ can certainly be simplified to a single number. Whether terms can be combined seems to depend on the relationship between them rather than whether they involve the same or different letters of the alphabet.

Sometimes, the distinction between a quantity like cost and a dimensionless 'number of ...' is a bit unclear. Does a distance *d include* the units (d = 10 metres) or is *d* 'the number of metres', in which case *d* is a pure number and the distance is '*d* metres'? (Note 2) In the latter case, *d* has no dimensions, so it isn't very obvious why I mustn't add, say, *d* and *t* (time), as then I'm just adding up two numbers. Perhaps the argument is that *mathematically* you could add them up, but the number you would obtain would have no real-world significance. But, perhaps it would, and you just have to think of the appropriate situation for it? (Note 3)

Returning to the original question about why we can't have 'distance + time', my difficulty is that the correct formula,

$$\text{speed} = \frac{\text{distance}}{\text{time}},$$

is no more obviously acceptable, from a dimensional point of view, than the incorrect one – *unless* you happen to be already familiar with units like 'metres per second'. Why do we say that 'metres *plus* seconds' is ridiculous but 'metres *divided by* seconds' is fine? Isn't this a bit odd? How can you possibly divide a metre by a second? They are in different units!

There doesn't seem to be a pure mathematical way to resolve this. Certainly, some quantities have useful properties in the world and others don't seem to (or perhaps their usefulness has not yet been realized?). But, in that case, there is nothing *mathematically* wrong with distance + time; it just probably isn't going to be very useful. However, it does seem that there is something fundamentally different about

multiplication/division
versus
addition/subtraction

in terms of what it makes sense to do.

To take a different example, what would happen if we multiplied a force by a distance? Someone might ask: How can you possibly multiply a newton by a metre? What on earth would you get? Well, you would get a 'newton-metre' – it just so happens that we have an alternative name for that, 'the joule', which sounds more respectable, but on the face of it it's still an odd combination of units. And there are far odder ones throughout science. There is certainly no rule against strange combinations of units, whether they are given single names or not. (Note 4)

Rather than relying on rules about what is 'allowed', we should be thinking about meaning. The Nobel-prize-winning physicist Richard Feynman (1992) described reviewing some school textbooks, in one of which some values were given for the temperature of different-coloured stars. He was mildly irritated that they referred to green and violet stars, which don't exist, but what really annoyed him was the subsequent question:

> John sees two blue stars and a red star. His father sees a green star, a violet star, and two yellow stars. What is the total temperature of the stars seen by John and his father? (Feynman, 1992, pp. 293–294)

He was outraged that there was 'no purpose whatsoever in adding the temperature of two stars', describing it as just 'a game to get you to add'. Just because something may be in the same units (here temperature) and, in some sense, *can* be added, does not mean that there is necessarily any purpose in doing so. Perhaps adding temperatures might make sense if you wanted to calculate the mean temperature for some reason (Note 5); by extension to that, I suppose you could say that multiplying the temperatures together could make sense if you wanted to calculate the *geometric* mean.

Dimensions are a powerful idea – I remember being astonished at school to see how knowledge of dimensions could enable you to work out the formula for something like the time period of a pendulum (up to a constant) – and even see that mass wasn't involved – which almost seemed too good to be true. But it seems that there is no rule from dimensions that can function as an alternative to thinking about what you are doing. Even when marking a mathematics test, does it really make sense to *add up* a pupil's mark on question 1 and their mark on question 2, if those questions happen to relate to completely different topics? By themselves, dimensions are a problematic way of deciding what we should or shouldn't be doing.

Notes

1. See **http://www.nhs.uk/Livewell/5ADAY/Pages/Why5ADAY.aspx**
2. The latter may be more common in mathematics and the former in science.
3. This reminds me of that sixth-form question: "If you integrate acceleration, you get velocity; if you integrate velocity, you get displacement. So what do you get if you integrate displacement?"
4. Torque is measured in newton-metres (N m), and this is dimensionally equivalent to the joule, the unit of work or energy. But this dimensional equivalence doesn't mean that it would be sensible to add a torque to an energy. In fact, rotational work is calculated by multiplying a torque by an angle, in radians, which is dimensionless.
5. It certainly would make sense to add temperatures if you were adding a starting temperature and a temperature increase to find a final temperature.

References

Feynman, R. P. 1992 *Surely You're Joking, Mr. Feynman! Adventures of a Curious Character*, Vintage, London.

Küchemann, D. 1978 'Children's Understanding of Numerical Variables', *Mathematics in School*, **7**, 4, pp. 23–26.

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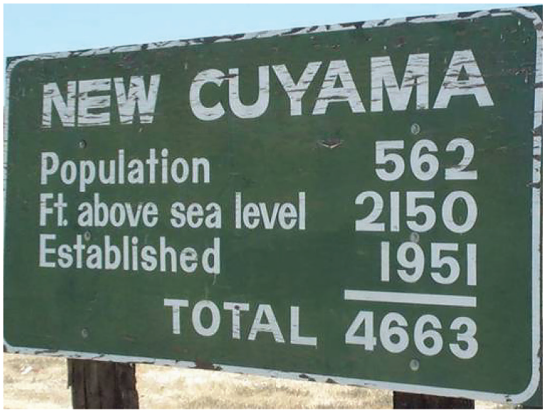

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