Teaching the multiplication tables multiplicatively

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argues for a better way of teaching the multiplication facts

Facts without any structure

Suppose, for some bizarre reason, I wanted you to memorise the 144 numbers in the table below.

6	10	9	10	5	10	10	9	18	19	22	22
6	9	11	9	9	8	12	11	10	16	22	22
3	6	6	5	5	5	7	6	6	10	16	19
4	8	11	9	9	9	10	10	9	6	10	18
5	7	10	9	5	10	8	9	10	6	11	9
5	8	9	11	10	8	9	8	10	7	12	10
3	6	8	10	6	9	8	10	9	5	8	10
4	3	7	6	10	6	10	5	9	5	9	5
4	5	6	7	6	10	11	9	9	5	9	10
5	3	4	6	7	8	9	10	11	6	11	9
3	4	3	5	3	6	8	7	8	6	9	10
3	3	5	4	4	3	5	5	4	3	6	6

Indeed, not only do I want you to memorise each number, but I want you to know *where* each number is in the table. So, I could say, "Row 2 from the bottom, column 8", and you could immediately respond, "7".

Unless you have an eidetic memory, I think you will agree this would be a lot of work. You might like to estimate how many hours you would need to devote to this task to achieve mastery – quite a lot, I would imagine.

The problem is that the numbers in the table don't seem to follow any obvious pattern. It would be like learning the first 144 digits of π, so that you could give any required digit when asked.

Actually the numbers in the table are not random. They are derived from the numbers in the standard multiplication square (with 1×1 in the bottom left corner). Each number in the table is the number of letters in the name of the number that would be in that position. For example, the bottom left square is 3, because $1 \times 1 = 1$, and the number 'one' has 3 letters.

Once you know that, the task becomes easy, because you can relate it to a familiar structure – that of the multiplication square. But without knowledge of the structure, you have to resort to rote memorisation, which is tedious, difficult and hard to retain.

Facts with the wrong structure

Of course, no teacher would expect children to learn their multiplication facts as 144 completely separate numbers without any structure. But I think often we give learners the 'wrong' structure – not mathematically wrong, but just not the most helpful structure for learning them. The mistake I think is to teach the multiplication tables *additively*, rather than *multiplicatively*.

12	24	36	48	60	72	84	96	108	120	132	144
11	22	33	44	55	66	77	88	99	110	121	132
10	20	30	40	50	60	70	80	90	100	110	120
9	18	27	36	45	54	63	72	81	90	99	108
8	16	24	32	40	48	56	64	72	80	88	96
7	14	21	28	35	42	49	56	63	70	77	84
6	12	18	24	30	36	42	48	54	60	66	72
5	10	15	20	25	30	35	40	45	50	55	60
4	8	12	16	20	24	28	32	36	40	44	48
3	6	9	12	15	18	21	24	27	30	33	36
2	4	6	8	10	12	14	16	18	20	22	24
1	2	3	4	5	6	7	8	9	10	11	12

The obvious way to handle the multiplication tables is table by table; i.e., column by column, or row by row, as shown by the shading in the table above.

To an adult, this seems logical and organised. Today we are doing the 7-times table. Next week, we will do the 8-times table. Everyone knows what is going on.

The trouble is that this is not the most mathematical way to carve up the multiplication table. Within any individual table, such as the 7-times table, the relationships are all *additive*. When you count 0, 7, 14, 21, 28, … you are *adding* 7 each time. This doesn't have much to do with multiplication.

Working this way is also very buggy, because as soon as you make one little slip, all the following numbers will be wrong. It can be heartbreaking to watch a child working out, say, 7×8 by counting up very slowly on their fingers 0, 7, 14, 22, … If you don't stop them there, they have slipped from the sequence $7n$ to the sequence $7n + 1$, and are doomed.

Through lengthy practice, they might eventually master their 7s. But then, next week, when we go on to the 8-times table, all the numbers will be just a little bit different, and the interference between last week and this week can be disastrous. Eventually, all the child learns is that there is a huge sea of numbers out there, and picking out the right one at the right time is a hazardous task. Numbers they are sure were right last week are suddenly wrong this week.

The reason this is so hard is that we are feeding learners with the *wrong structure*. The row-by-row and column-by-column patterns are not actually very useful.

Facts with the right structure

In my opinion, teaching the multiplication facts table by table is suboptimal. We need to go with the flow of the real structure of the grid, rather than fighting against it. It is a *multiplication* table, so the focus needs to be *multiplicative* (Foster, 2022).

How can this work in practice?

The first thing is to stop running along the rows and down the columns. My favourite way to subvert this with learners who have previously struggled with their tables is to work on the *squares*.

12	24	36	48	60	72	84	96	108	120	132	144
11	22	33	44	55	66	77	88	99	110	121	132
10	20	30	40	50	60	70	80	90	100	110	120
9	18	27	36	45	54	63	72	81	90	99	108
8	16	24	32	40	48	56	64	72	80	88	96
7	14	21	28	35	42	49	56	63	70	77	84
6	12	18	24	30	36	42	48	54	60	66	72
5	10	15	20	25	30	35	40	45	50	55	60
4	8	12	16	20	24	28	32	36	40	44	48
3	6	9	12	15	18	21	24	27	30	33	36
2	4	6	8	10	12	14	16	18	20	22	24
1	2	3	4	5	6	7	8	9	10	11	12

The square numbers march confidently right through the middle of all these tables. For a teacher who is used to going table by table, the sequence of squares is annoying, because each square is in a different table. But for a learner who has had limited success in the past with tables, the squares are extremely powerful.

Any tricky fact you might need won't be far away from a square number. If you were being sent to a desert island and had to choose just 12 multiplication facts to take with you, I would choose the squares. (As an aside, for learners who quickly master up to 12×12, learning the squares up to 20^2 potentially *doubles* the number of multiplication facts they have reasonably easy access to, and is a high-leverage move.)

With students at secondary school who by Year 10 still 'do not know their tables', I have sometimes reluctantly settled for just knowing the squares up to 12^2. You can do a lot with just that.

What next after the squares?

We can get a very long way just by developing learners' skills with *doubling*. The point with the 2-times table is not so much to be able to add 2, as 0, 2, 4, 6, 8, … but to be able to *double any given number*. I say 'Double 3', you respond with '6'; I say 'Double double 10', you respond with '40'.

I wouldn't do anything else until learners are really good at doubling any given number, first without 'carries' and then with. Doubling is a valuable skill in its own right. But once they can do this, we can get almost all the way through the remaining tables without too much trouble.

For example, once learners are confident with $2 \times 3 = 6$, by doubling this they can obtain

$$4 \times 3 = 12$$
$$2 \times 6 = 12,$$

and then by 'double double' they can get

$$4 \times 6 = 24$$
$$2 \times 12 = 24$$
$$8 \times 3 = 24.$$

No number appears more often in the multiplication square than 24, and this is no fluke – there are no coincidences in the multiplication square! The reason for all these 24s is apparent when we look at the tables multiplicatively, rather than additively.

I would do all of this calculating orally, so learners are becoming skilled at doubling mentally and holding the numbers in their heads. When working in this way, I frequently find learners will happily slip outside the limits of the 12×12 multiplication square, offering facts such as

$$16 \times 3 = 48$$

through doubling $8 \times 3 = 24$, perhaps without even realising they are not 'supposed' to know that one!

The doubling sequence $3 \rightarrow 6 \rightarrow 12 \rightarrow 24 \rightarrow 48$ doesn't involve any 'carries', and even very young children can get very confident at doing this. We need just one carry to get the final value of $8 \times 12 = 96$.

12	24	36	48	60	72	84	96	108	120	132	144
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10	20	30	40	50	60	70	80	90	100	110	120
9	18	27	36	45	54	63	72	81	90	99	108
8	16	24	32	40	48	56	64	72	80	88	96
7	14	21	28	35	42	49	56	63	70	77	84
6	12	18	24	30	36	42	48	54	60	66	72
5	10	15	20	25	30	35	40	45	50	55	60
4	8	12	16	20	24	28	32	36	40	44	48
3	6	9	12	15	18	21	24	27	30	33	36
2	4	6	8	10	12	14	16	18	20	22	24
1	2	3	4	5	6	7	8	9	10	11	12

If we assume that by this point learners have also mastered their 0s, 1s, 5s and 10s, then, with doubling as our superpower, we are in a very healthy situation, with all the green facts above available almost instantly.

Learners will now be used to the fact that when we double *either* of the factors, the product doubles – and if we double *both* of the factors, the product double doubles!

The other thing we can do with a product is to *double* one factor and *halve* the other, and then the product doesn't change. This enables us to obtain something like 8×3 directly from 4×6, without having to go back to 2×3.

Now, if learners master the 11s (the easy double-digits of 10 lots and another one) and the 9s (10 lots *minus* one lot), then all that remains are three missing tables.

12	24	36	48	60	72	84	96	108	120	132	144
11	22	33	44	55	66	77	88	99	110	121	132
10	20	30	40	50	60	70	80	90	100	110	120
9	18	27	36	45	54	63	72	81	90	99	108
8	16	24	32	40	48	56	64	72	80	88	96
7	14	21	28	35	42	49	56	63	70	77	84
6	12	18	24	30	36	42	48	54	60	66	72
5	10	15	20	25	30	35	40	45	50	55	60
4	8	12	16	20	24	28	32	36	40	44	48
3	6	9	12	15	18	21	24	27	30	33	36
2	4	6	8	10	12	14	16	18	20	22	24
1	2	3	4	5	6	7	8	9	10	11	12

These all belong to the same family of 7s. I think it is worth just remembering $3 \times 7 = 21$, because then the others come by easy doubling:

$$6 \times 7 = 42$$
$$12 \times 7 = 84.$$

Supporting learners

It is all very well figuring this out in your arm chair. But where it really shows is of course when learners get stuck. Working multiplicatively means you offer quite a different kind of help than what teachers might typically do.

For example, suppose a child is stuck on the notorious 7×8. Typically a teacher might ask which other multiples in the 7- or 8-times tables the child can remember. Quite often, if they don't know 7×8, they are not very confident with anything nearby. They might say, "Well, I know that 7 × 7 is 42", but since that is not true it doesn't help. Even if they did know 7 × 7 =49, they might be quite unsure whether they needed to add on another 7 or another 8.

The teacher who is working multiplicatively will respond very differently: "Do you know 7 × 2?" This may seem miles away, but it is very likely they will know this, or be able to find it. Now they just have to double twice: 14, 28, 56.

Later, the teacher will just need to ask, "Which fact can you get 7 × 8 from?", and the learner will go back to 7 × 2 for themselves. Later still, the learner will manage all this themselves, often extremely quickly. And eventually they will have internalised these relationships so much that they 'just know', and no cognitive resources are required at all.

However, by building multiplicatively, should the learner ever have their mind go blank, in the stress of the moment, or have a crisis of confidence ("Is 7 × 8 really 56?"), they will be able to quickly rebuild up from easy foundations to obtain the answer.

Reference

Foster, C. (2022). Learning times tables through systematic connections. **https://www.foster77.co.uk/Learning%20Times%20Tables%20Through%20Systematic%20Connections.pdf**

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Beyond the 100 square?

Most teachers and primary pupils are familiar with the 0-99 (or 1-100) square as a support for number understanding, but why not try this extended square.

-33	-32	-31	-30	-29	-28	-27	-26	-25	-24	-23	-22	-21	-20	-19	-18
-23	-22	-21	-20	-19	-18	-17	-16	-15	-14	-13	-12	-11	-10	-9	-8
-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2
-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12
7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42
37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52
47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62
57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72
67	68	69	70	71	72	73	74	75	76	77	78	79	80	81	82
77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92
87	88	89	90	91	92	93	94	95	96	97	98	99	100	101	102
97	98	99	100	101	102	103	104	105	106	107	108	109	110	111	112
107	108	109	110	111	112	113	114	115	116	117	118	119	120	121	122
117	118	119	120	121	122	123	124	125	126	127	128	129	130	131	132

What can children see in the extended patterns?

Is this useful to introduce negative numbers?

Can they extend it further in each direction?

Ray Huntley is mostly retired but loves collecting and sharing new teaching ideas.