HALF of the SUM of the OTHERS

Colin Foster

Look at these five numbers:

$$3, 10, 12, 15, 20.$$

Do you see anything special about them?

Which one is half of the sum of the others?

Which one is one-third of the sum of the others?

Which one is one-quarter of the sum of the others?

Which one is one-fifth of the sum of the others?

Can you make up a set of numbers 'like this'? You will need to decide what 'like this' could mean.

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This task fits into the genre of 'Create something nice', and the way I would typically set this up in the classroom would be:

1. Present my creation.
2. Help students to appreciate what is 'nice' about it.
3. Invite students to make their own examples.

This is hard to do by trial and error, but a bit of algebra really helps. Below, it looks heavier than it is because there are so many unknowns, but it just involves adding, subtracting, multiplying and dividing, and students familiar with adding and subtracting equations when solving linear simultaneous equations by elimination should know everything they need to have a go.

Suppose we want a set of 6 distinct numbers:

$$a > b > c > d > e > f.$$

Then we have:

$$a = \frac{b+c+d+e+f}{2}$$

which is equivalent to

$$-2a + b + c + d + e + f = 0.$$

Similarly,

$$b = \frac{a+c+d+e+f}{3}$$

which is equivalent to

$$a - 3b + c + d + e + f = 0.$$

So we can see that the whole set of equations is going to be:

$$-2a + b + c + d + e + f = 0 \qquad (1)$$

$$a - 3b + c + d + e + f = 0 \qquad (2)$$

$$a + b - 4c + d + e + f = 0 \qquad (3)$$

$$a + b + c - 5d + e + f = 0 \qquad (4)$$

$$a + b + c + d - 6e + f = 0 \qquad (5)$$

Subtracting (1) from (2), we obtain

$$3a - 4b = 0,$$

and, continuing in this way, we obtain:

$$4b - 5c = 0$$

$$5c - 6d = 0$$

$$6d - 7e = 0.$$

This means that we have a chain of numbers in which each one, except the first and last, is obtained by multiplying the previous one by a fraction multiplier:

$$\begin{array}{ccccccccccc} & \times\frac{3}{4} & & \times\frac{4}{5} & & \times\frac{5}{6} & & \times\frac{6}{7} & & & \\ a & > & b & > & c & > & d & > & e & > & f \end{array}$$

and the last one, f, must be given by

$$3b - a - c - d - e. \text{ [1]}$$

So, the set of numbers that we want must be some constant multiple of the set:

$$1, \frac{3}{4}, \frac{3}{5}, \frac{3}{6}, \frac{3}{7}, \left(\left(3\times\frac{3}{4}\right) - 1 - \frac{3}{5} - \frac{3}{6} - \frac{3}{7}\right)$$

which is:

$$1, \frac{3}{4}, \frac{3}{5}, \frac{3}{6}, \frac{3}{7}, -\frac{39}{140}.$$

Multiplying up by 140 gives

$$140, 105, 84, 70, 60, -39$$

which are all integers – but *not all positive*.

These numbers work, because they sum to 420, and

$$140 = \frac{420-140}{2},$$

$$105 = \frac{420-105}{3},$$

$$84 = \frac{420-84}{4},$$

$$70 = \frac{420-70}{5},$$

$$60 = \frac{420-60}{6}$$, as required.

But, why do we have a *negative* number as part of our set? Can we avoid this? In the original example, where we had five positive numbers, following the algebra would have given us:

$$1, \frac{3}{4}, \frac{3}{5}, \frac{3}{6}, \left(\left(3\times\frac{3}{4}\right)-1-\frac{3}{5}-\frac{3}{6}\right)$$

and the final number here is equal to, $\frac{3}{20}$ which is positive, which means that we end up with

$$1, \frac{3}{4}, \frac{3}{5}, \frac{3}{6}, \frac{3}{20}$$

and multiplying up by 20 gives us the set of numbers 20,15,12,10,3 that we started with. However, we can see that with six or more numbers the final number is going to be negative.

For completeness, the other smallest possible sets of integers that begin this family are shown in the table below.

Number puzzles like this need not consume a lot of classroom time, but they generate some useful arithmetic practice, provide a natural need for some simple algebra, and can be very enjoyable.

Note

[1] We are assuming that *f* necessarily comes out as the smallest one, which it does in this case.

Author:
Dr Colin Foster, Mathematics Education Centre, Schofield Building, Loughborough University, Loughborough LE11 3TU.
Email:c@foster77.co.uk
Website: www.foster77.co.uk

Number of numbers	*Numbers*
3	4, 3, 5
4	20, 15, 12, 13
5	20, 15, 12, 10, 3
6	140, 105, 84, 70, 60, –39
7	280, 210, 168, 140, 120, 105, –183
8	840, 630, 504, 420, 360, 315, 280, –829
9	840, 630, 504, 420, 360, 315, 280, 252, –1081
10	27,720, 20,790, 16,632, 13,860, 11,880, 10,395, 9240, 8316, 7560, –43,233
11	27,720, 20,790, 16,632, 13,860, 11,880, 10,395, 9240, 8316, 7560, 6930, –50,163

MATHEMATICIANS ON STAMPS (2)

Stamps featuring Zu Chongzhi, Leonhard Euler and Augustin Cauchy.