Tailoring the Examples to the Method

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Imagine a teacher who needs to teach a particular topic – say, addition and subtraction of directed (positive and negative) numbers. Suppose they find a worksheet online, or a set of exercises in a textbook. Great. Now all they need to do is to think about how they are going to explain to the pupils how to do them (see French, 2001). Maybe they think about that now, or maybe they see how the mood takes them on the day. Or maybe they have multiple possible approaches up their sleeve, and they will offer more than one of these approaches to the class, and perhaps just see which seems to work best for each pupil.

I think this is problematic, because it seems to me that it is impossible to construct a set of questions on this, of progressing difficulty, without reference to the *method* that the pupils are going to use to answer them. For example, which of these calculations do you expect a pupil to find harder to do?

(a) 5 + (–2)

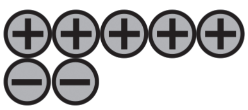
(b) (–5) – (–2)

If your model is 'vectors on a number line' (see Mattock, 2019), then (a) seems easier than (b). For (a), you find 'positive 5' on the number line, and then you have to 'add on a negative two', which moves you backwards to 3, so the answer is 3. Whereas (b) is much more complicated, because you have to start at 'negative 5' and then do some kind of manoeuvre to represent the subtraction of a negative number, such as facing to the left but walking backwards, to arrive at –3. So, (a) seems to be easier than (b). This is probably no surprise, since we all know that 'double negatives' are difficult (e.g., see Foster, 2015).

But, if you are using a different model, such as double-sided counters [1] (see Foster, 2013, 2015), then everything changes. Now, calculation (b) suddenly becomes easy: you have five negative counters and you 'take away' two of them:

so you are left with 'negative 3'.

In this model, this is *easier* than (a), where you would have five positive counters and then have to 'add' to this 'two negatives':



and then do some cancelling out to end up with 'positive 3'.

So, the order of difficulty *reverses* depending on the model you are using; it is not an absolute to say that one calculation is harder than another one. This means that a set of exercises of 'increasing difficulty' has embedded within it assumptions about how the pupils are going to go about it. Indeed, 'difficulty' is just one dimension to consider in a set of questions, and, in general, a set that is optimised for one method will not be optimal for another. So, we shouldn't be able to switch between different sets of exercises on the same topic without careful consideration.

In fact, even the way in which the author chooses to *write* these calculations communicates something about how they are being conceptualised. We could have:

5 + (–2)

or (+5) + (–2)

or +5 + –2

or $^{+}5 + {}^{-}2$

or they might be represented in pictorial form as counters or number lines. Each gives a different sense of what the calculation might mean, and there are often clashes here. In one classroom, a teacher is telling pupils that (+5) + (–2) can be 'simplified' into 5 – 2, because 'a plus and a minus next to each other makes a minus', and, in the classroom next door, another teacher is telling pupils that what subtraction *really* means is an addition of the additive inverse, so they need to think of 5 – 2 as (+5) + (–2) (see McCourt, 2019, p. 168). These teachers are pushing in opposite directions. If pupils are working the calculations (a) and (b) above at this kind of symbolic level, then 'simplifying' (+5) + (–2) into 5 – 2 is obviously easier than exchanging (–5) – (–2) for –5 + 2, and, even then, deciding that this is –3, and not +3 or –7, is still likely to be hard. So, here, we are back to (a) being easier than (b).

The choice of model is actually a bit more subtle than just 'number lines or counters', as it depends in each case on exactly how these different devices are being used. For example, a different way to use the number line for (b), rather than considering vector journeys, is to interpret the question in terms of 'difference'. So, here we would mark the two points –5 and –2 and notice that there is a gap of 3 units between them, and then, because of the order in the calculation, we would describe this difference as '*negative* 3'. Using the number line in this way, it is perhaps easier to see that the answer is going to be ±3 (rather than ±7), but harder to be sure about the sign than when taking the vector approach.

Whatever the pros and cons of these different approaches in terms of pupils' understanding, my point is that calculations that are easier in one method may be harder in another. Working recently with colleagues in Japan, and exploring how mathematics textbooks designed there take very careful account of the particular methods that they anticipate that pupils will use (see Seino & Foster, 2019), has made me question the wisdom of pulling resources from all kinds of different places and hoping that they will somehow mesh. Differently-sourced materials, designed from different standpoints and assumptions, ought to seriously clash. Quickly photocopying a page of questions from a textbook, and then using them without reference to how the book had set them up, should be a disaster. If the exercises were well constructed, a teacher should be able to tell, just from looking carefully at them, how the author was expecting the pupils to go about them. I doubt that many sets of exercises used in classrooms in the UK, from any source, would pass that test. We get away with it because, in general, our resources are not very carefully designed, so in practice it makes little difference. But that should be a challenge to develop coherent sets of tasks that reflect principled choices about the ways in which important ideas will be introduced and sequenced.

Note

1. Yes, I know that all counters are double-sided, but by this I mean small circular counters that have a different colour on each side.

Acknowledgement

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References

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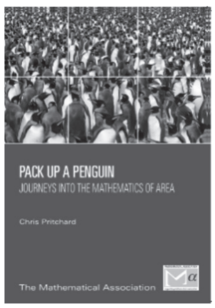
Note on the Cover Images

We are fortunate in having two stunning cover images for this issue. They were taken by my daughter, Dr Ceri Pritchard, in Siracusa, Sicily and they celebrate the life and work of the city's most famous resident, the mathematical genius, Archimedes.

The π symbol sits in the middle of a roundabout and is an allusion to Archimedes' use of proof by contradiction to find very narrow bounds for its value. This provided mathematicians with the tools to calculate the area of a circle to greater accuracy and of course, much, much more. This is one of two aspects of Archimedes' mathematics (the other being his method for finding the area between a parabola and a straight line) that I explore in one of the chapters of my new book *Pack up a Penguin: Journeys into the Mathematics of Area*, which is available from The Mathematical Association (www.m-a.org.uk).



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