Appreciating the second-derivative test

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When teaching calculus, I find that students often seem to dislike the second-derivative test for classifying stationary points. They accept the need to find the first derivative, in order to discover the number and location of any stationary points, by finding places where the first derivative is zero. But then differentiating for a second time seems like a lot of work, and they worry that it will all be in vain if the second derivative turns out also to be zero at a stationary point, meaning that the test is inconclusive. For example, if $f(x) = (x-3)^4$, then the first derivative, $f'(x) = 4(x-3)^3$, which is zero only when $x = 3$, so we have a single stationary point at (3, 0). If we then calculate $f''(x) = 12(x-3)^2$, we find that $f''(3) = 0$ also, so the second-derivative test tells us nothing.

At this point, students are normally expected to give up with derivative tests and determine the nature of the stationary points by evaluating f' a little to the left of the stationary point and a little to the right of the stationary point. They then conclude the nature of the stationary point based on whether f' is negative (f decreasing) before and positive (f increasing) afterwards (a local minimum) or f' is positive before and negative afterwards (a local maximum). Having f' positive before and after, or negative before and after, correspond respectively to an increasing point of inflection and a decreasing point of inflection.

Indeed, I find that students often prefer merely to evaluate f itself, either side of the stationary point. This seems sensible, since there is always a chance of an error in f', and going back to the original, given function seems safer. If both function values are greater than they are at the stationary point, then we have a local minimum, and if they are both less then we have a local maximum. Less to the left and greater to the right corresponds to an increasing point of inflection, and greater to the left and less to the right corresponds to a decreasing point of inflection.

Students often seem to be confused about the 'small distance' that needs to be used for the deviation either side of the stationary point. I find that they will frequently use differences such as 0.1 or 0.01, on the grounds that these are 'small numbers'. But small relative to what? I think that this behaviour may derive from how differentiation has been presented originally, in terms of tiny δx increments, and students worry that their answers will be invalid if they use a value that is 'too large'. What they often seem not to realise is that the sign of f' cannot change between consecutive stationary points, so any convenient (e.g. integer) value within that interval will give the same result (Note 1). For our function, $f'(x) > 0$ for all $x > 3$ and $f'(x) < 0$ for all $x < 3$, so there is no need to be using a calculator to evaluate $f'(2.999)$ and $f'(3.001)$ when $f'(0)$ and $f'(4)$, say, would be just as informative. The same goes for checking whether f itself is greater or less either side of the stationary point than it is at the stationary point.

Another reason that students seem to dislike the second-derivative test, even when $f''(x) \neq 0$, is that identifying a local maximum with $f''(x) < 0$ and a local minimum with $f''(x) > 0$ somehow feels psychologically the wrong way round. ("Maximum is negative and minimum is positive?") We may explain that a local minimum involves a function whose gradient increases through zero (from negative – downhill – to positive – uphill), and so positive gradient-of-the-gradient corresponds to a local minimum. However, students still often seem uneasy with this, and worry that they might have it the wrong way round, which I think also discourages them from using the second-derivative test.

I will now describe two ways in which I try to encourage a more positive appreciation of second derivatives.

Higher derivatives

Perhaps students would be more comfortable with the second-derivative test if they knew that they could continue with the process in the case where f'' is zero at a stationary point. Rather than giving up when f'' is zero, it is usually possible to arrive at a conclusion if we just continue differentiating. For our example, with $f(x) = (x-3)^4$, we can continue differentiating past $f''(x)$ to higher derivatives, if we wish:

$$\begin{aligned} f(x) &= (x-3)^4 \\ f'(x) &= 4(x-3)^3 \\ f''(x) &= 12(x-3)^2 \\ f'''(x) &= 24(x-3) \\ f^{\text{IV}}(x) &= 24. \end{aligned}$$

If the first non-zero derivative (at the stationary point) is an even-numbered derivative, such as here, where the fourth derivative is the first one that is not zero at the stationary point, then we can conclude exactly as we would for the second derivative. Because the fourth derivative is positive, we have a local minimum. If it were negative, we would have a local maximum. (The zero – inconclusive – case doesn't apply, because we are focusing on a non-zero derivative.)

If the first time that we get a non-zero derivative at the stationary point is for an odd-numbered derivative, then if this value is positive we have an increasing point of inflection, and if it is negative then we have a decreasing point of inflection. We can see an example of such a function if we take our first derivative above, $4(x-3)^3$, and call it $g(x)$.

Now,

$$g(x) = 4(x-3)^3$$
$$g'(x) = 12(x-3)^2$$
$$g''(x) = 24(x-3)$$
$$g'''(x) = 24.$$

Starting with g, our first non-zero derivative is the third derivative, g''', and it is positive, telling us that the function g has an increasing point of inflection at (3, 0).

The extra complexity of higher derivatives seems worthwhile to complete the story and avoid leaving students stuck when the second derivative turns out to be zero at a stationary point (Note 2). Indeed, we can think of the second-derivative test itself as being motivated by discovering the first derivative to be zero: the principle is that we keep differentiating until we find a derivative that isn't zero.

Why does this work? The positive fourth derivative that we obtained for f means that the third derivative is increasing. Since the third derivative is zero at the stationary point, the third derivative is increasing *through zero* (i.e. from negative to positive). Since the third derivative is the gradient of the second derivative, this means that the second derivative must be a minimum at the point (i.e., a function with gradient increasing through zero). Since the second derivative is also zero at the stationary point, it must be positive to the left and also positive to the right, meaning that the first derivative is increasing on both sides of the stationary point. Since the first derivative is also zero at the stationary point, it must be increasing through zero, meaning that the original function has negative gradient on the left and positive gradient on the right, and is therefore a minimum. You can always work through this reasoning, derivative by derivative, for as long as needed, to see why these results follow, and it doesn't require any algebra.

Concavity

My other attempt to help students become more comfortable with second derivatives has been to teach about concavity before meeting the second-derivative test. We will have previously spent lots of time on the big question, "What does f' tell us about f?" We will have found that the sign of f' tells us whether f is increasing or decreasing, and that the magnitude of f' tells us how quickly it is doing so. After this, the next big question is, "What does f'' tell us about f?", and this is harder to think about, because we have to use f' as an intermediary between f'' and f. If f'' is positive, then f' is increasing, and if f' is increasing then the gradient of f is either becoming more positive or less negative. So f'' being positive corresponds to f being concave upwards. I try to do all of this without suggesting that f' necessarily has to be zero anywhere. The second derivative is informative *everywhere*, not just at stationary points. A similar argument shows that f'' being negative corresponds to f being concave downwards. Once students are confident in interpreting concavity, the second-derivative test is just an application of this, in the situation in which we are interested in concavity at places where f' happens to be zero.

Notes

1. We are of course assuming throughout that we have nice, continuous and repeatedly-differentiable functions, of the kind typically encountered in school!
2. In these examples, the first non-zero derivative happens to be a constant (i.e. independent of x), but of course that is not essential. What matters is that the derivative is non-zero *at the stationary point in question*.

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