COMPASS CONSTRUCTIONS ARE **NOT** METHODS

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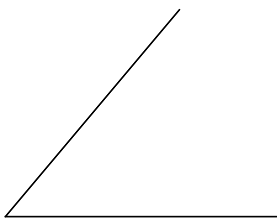
Here's a problem I've seen when teaching compass constructions. The teacher acts as though 'constructing an angle bisector' is a kind of 'method' – i.e., a way of achieving some useful goal – in the same way that something like 'completing the square' is a method for solving a quadratic equation. One day you might want to construct an angle that is equal to *half* of an angle you are given, and the angle bisector construction is the best or perhaps the most accurate method to do it. But I think this is the wrong way to think about what compass constructions are and why we teach them.

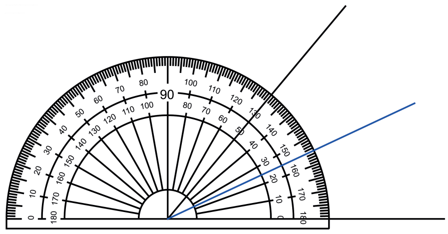
The reason this doesn't work for compass constructions is that they don't appear to enable students to do anything that they can't already do more easily. When we teach a method, we often start by giving students a problem that someone might want to solve and letting them struggle to do it *without* the method that you are about to teach. For example, you might present a quadratic equation that cannot be solved by factorisation and ask students to try to solve it by naïve brute force, perhaps by substituting in possible values. Students will find this inefficient, and maybe ineffective, and so we go on to introduce a method like completing the square as a better way (Foster, 2022). However, this sort of pedagogical approach seems doomed to fail for compass constructions.

Suppose you present students with two line segments meeting at a point:

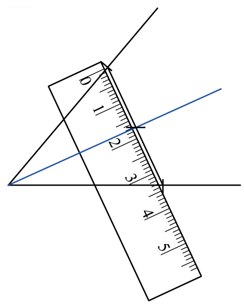


and ask them to think about how they would *bisect* the angle. They will have no trouble at all: "Get out a protractor, measure the angle, halve it, mark the halved angle, draw the line. Easy!" In Dan Meyer's (2015) terms, there is no 'headache' for which the technique you want to teach is the 'aspirin'.

To proceed, you then have to tell the students that, well, you're not allowed to use a protractor, for some mysterious reason, even though it seems the ideal tool for the job: "Imagine you had forgotten to bring it", you might say. Not an unlikely scenario, perhaps – except that it becomes a bit stretched when you have to admit that you do nevertheless have your pair of compasses with you, even though they probably live in the same little box as the missing-in-action protractor.



OK, say the students: We're not allowed a protractor – no problem. All we need to bisect this angle is *a ruler*: "Measure 4 cm along each line segment and put a mark. Join up these marks with a new line segment. Measure the length of this line segment and halve it. Put a little mark and join it to the vertex. Easy!"

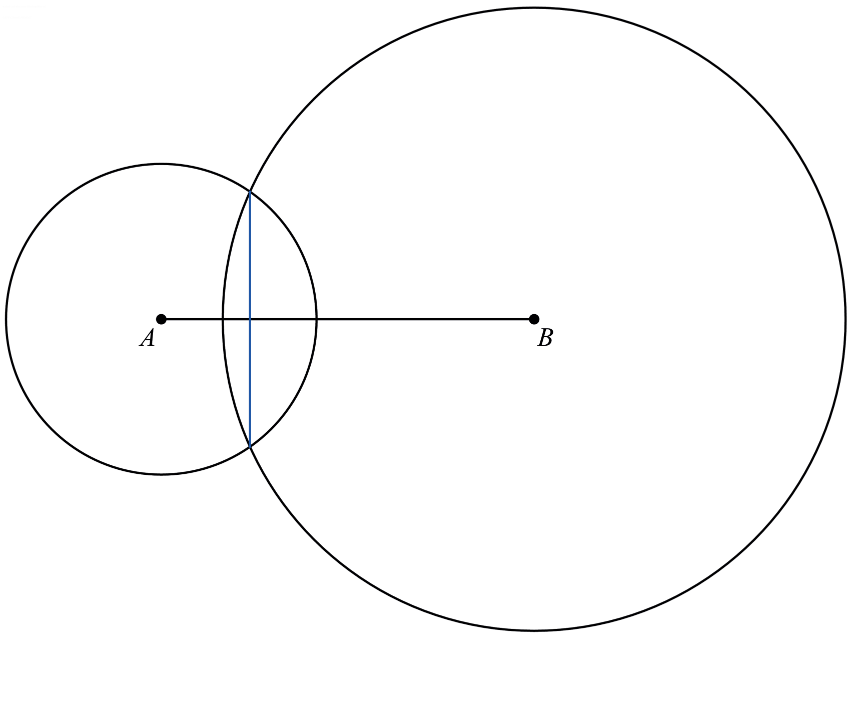


Now we have to say, "Sorry. Your ruler may not exactly be missing, but imagine that it's really old and all the markings on it have rubbed off, so it's just a 'straight edge'." (Note 1) It seems that we are getting into the far reaches of far-fetched here. Why all these arbitrary constraints? In an examination, students will have full access to protractors and unfaded rulers. If they can use them to *check* their constructions, why not use them to *do* the constructions in the first place (Note 2)? Sometimes the argument is given that constructing an angle bisector with compasses and straight edge is much more accurate than measuring the half-angle with a protractor, but this seems very unlikely to be true.

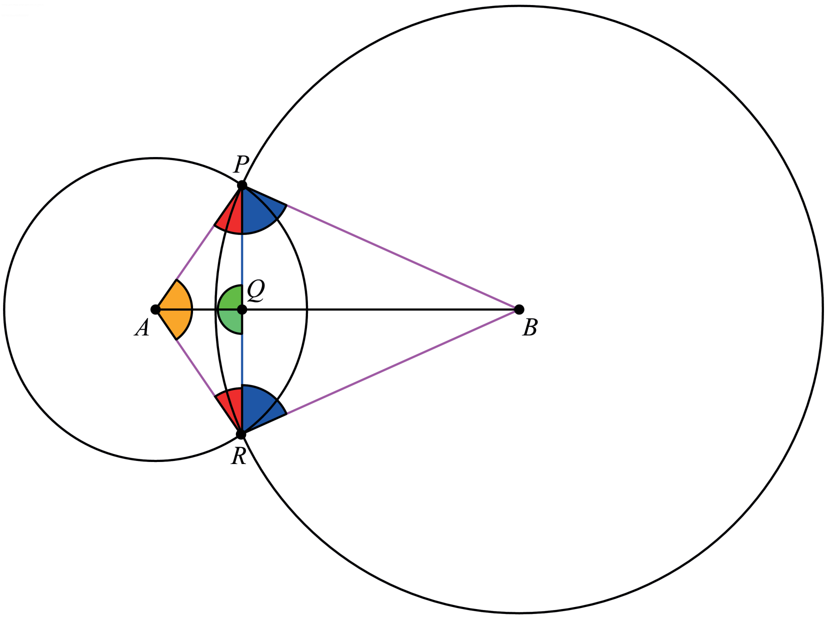
Perhaps, instead, we pose constructions as a 'What would you do if…?' kind of scenario, where the equipment constraints are not there for any realistic reason but purely to pose an intriguing puzzle. Just *imagine* that all you had was a straight edge and ruler – would there be any way to create a right angle, or even make a complete protractor, if you weren't allowed to use an already-existing one (Foster, 2006)? Even here, I have found that this doesn't usually work too well. The obvious way to bisect an angle, after all, is merely to *fold* the paper, so that the two lines lie on top of each other, and then the fold line is the angle bisector – simple! Similarly, with perpendicular bisectors, and assuming that that handy set square is also missing in action, simply folding the ends of the line segment onto each other will produce the desired perpendicular bisector in an instant. I find paper folding a useful initial activity to do with students, just to make clear what we *mean* by these different bisectors, before we start constructing them.

It seems to me that the problem is that compass constructions are not in the curriculum because they are a handy method to use when you happen to have lost your protractor and the markings have been unaccountably erased from your ruler – and you're drawing on an unfoldable surface, such as the wall of a building or a classroom whiteboard (Note 3). The classical, traditional compass-and-straight-edge constructions are in the curriculum (here in England) because they provide access to geometrical reasoning from basic principles that is really important in its own right. The constructions are *exact in principle*, even if, in practice, with a wobbly pair of compasses, and multiple steps, measuring an angle and halving is likely to be more accurate. I think students miss the beauty of compass constructions if they see them as a method to get a line in a certain place, or are misled into thinking that they are somehow more accurate than using measuring devices designed for exactly that purpose. An emphasis on accuracy doesn't help in this topic – I don't really care whether students are accurate to within $\pm 1°$ or $\pm 2°$ with their constructions. What I care about is that they can see and understand *why and how* these constructions produce – in principle with perfect accuracy – the lines that they are supposed to. In fact, *sketching* the constructions freehand, rather than making them accurately, ought to be just as valid. The skill of accurate 'technical drawing' shouldn't be relevant to the modern study of mathematics. If you teach the 'how-to' of the constructions, but "don't prove them", then I think there is no point to the topic whatsoever.

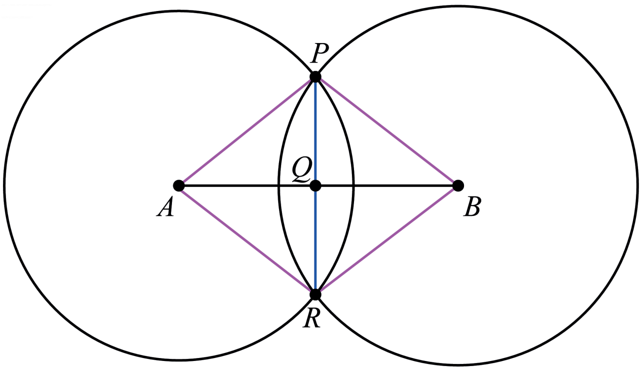
I have sometimes heard people say that the big idea of compass constructions is that they are 'all about isosceles triangles' or 'all about rhombuses'. But I think the way I would now approach them is via *overlapping circles* (Note 4), using dynamic geometry software and 'people maths' (Foster, 2015). The big starting question for me is: *What happens when two circles overlap?* I would like students to use mental visualisation to think through the four possibilities: no overlap, touching at a point, intersecting at two points, completely coincident. In the case where the circles intersect at two points, the blue line segment below joining those two points will always be perpendicular to the line segment *AB* joining the centres of the circles – and this is the case regardless of whether the two circles have the same radius or not.



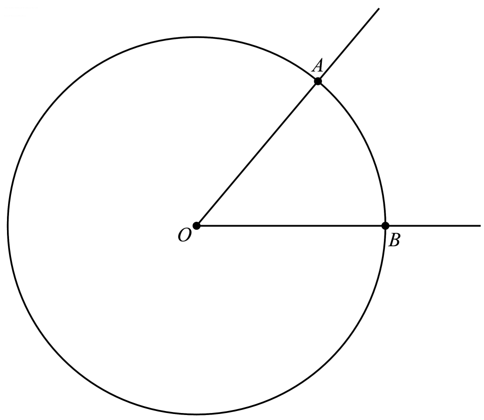
To prove this, we just need to show that triangles *APB* and *ARB* below are congruent (SAS, since the red angles and the blue angles are both pairs of equal base angles in isosceles triangles). It then follows that the orange angles are equal, and therefore that the triangles *APQ* and *ARQ* are congruent (SAS), meaning that the green angles must be equal to each other, and, since these add up to a straight line, each of them must be a right angle.



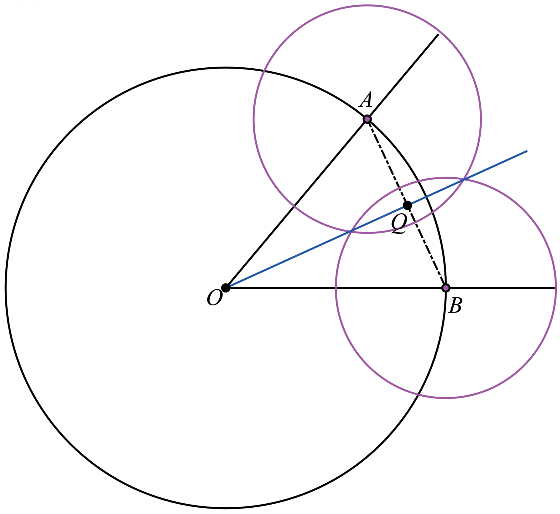
I think it's helpful to separate out the *perpendicular* aspect and the *bisector* aspect. In the special case where the two circles have *the same radius*, the perpendicular line will also *bisect* the line segment *AB*, and so we have a construction for a perpendicular *bisector*. In the diagram below, we have congruent isosceles triangles *APR* and *BRP* (SSS), meaning that triangles *APQ* and *BPQ* are also congruent (SAS), so that corresponding sides *AQ* and *BQ* are equal (i.e., *Q* bisects *AB*) and corresponding angles *AQP* and *BQP* are equal. And, since these two angles add up to a straight angle, each of them must be a right angle. This approach emphasises why the compasses must be opened the *same amount* for both arcs if we want to *bisect* the line segment, but if we just want any perpendicular line then it doesn't matter (Note 5).



For me, the perpendicular bisector naturally comes first, because an angle bisector can be viewed as a specific application of this. To bisect the angle we started with, we first use compasses to create points *A* and *B*, equidistant from the vertex *O*. (This is exactly the same move that students would make if they had a ruler – they would measure equal distances along both lines.)

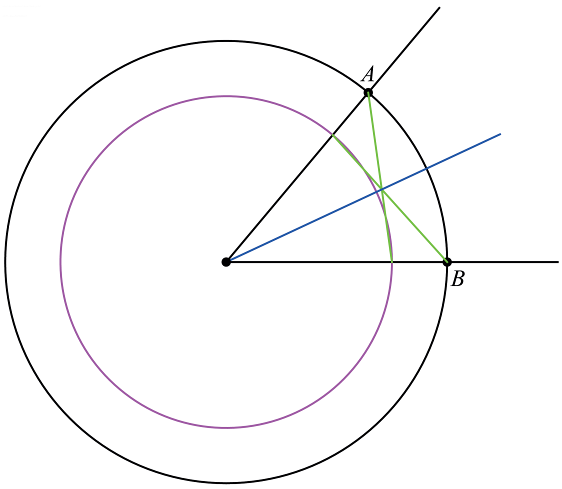


Then we construct the *perpendicular* bisector of *AB*, exactly as before, by drawing a circle, centred on each point, with equal radius (greater than half the distance *AB*), and joining up the intersection points.

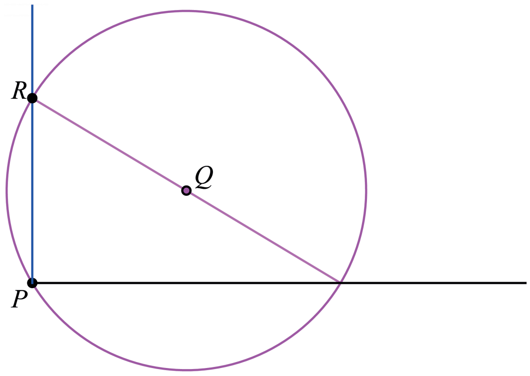


Now, we just have to prove that the angle bisector *OQ* (where *Q* is on *AB*) is the same line as this perpendicular bisector we have just constructed. We can do that by showing that triangles *AOQ* and *BOQ* are congruent (SAS) and therefore that corresponding sides *AQ* and *BQ* are equal, and corresponding angles *AQO* and *BQO* are equal – and supplementary – and so both are right angles.

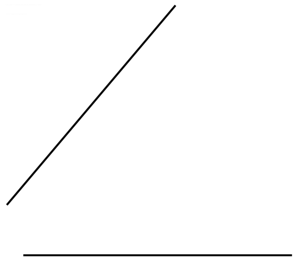
There are other ways to construct an angle bisector, as described in Southall (2020), such as the one below, which I do agree is easier *to do*, and so also possibly more accurate, but I find this quite a bit harder to prove.



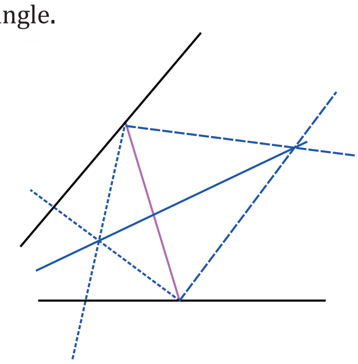
There are lots of interesting variations on the basic constructions when there are additional constraints. For example, how do you create a perpendicular line from a given point *P* lying *on* the given line? The classical construction involves creating points equal distances either side of *P* and then constructing the perpendicular bisector of the line segment joining these new points. But this isn't the easiest way, and, if *P* happened to be at the *end* of a line segment, and there was no space to extend it (e.g., *P* was right on the edge of a piece of paper), this wouldn't work. An easy alternative is to choose any point *Q* that is *not* on the line and draw a circle centred on *Q* passing through *P*. Then, draw a line from the point where this circle intersects the given line, passing through *Q*, until it intersects the circle again, at *R*. Then *RP* is the required perpendicular line, and this construction is probably also easier even when *P* isn't in such an awkward position (Engel, 1998, p. 316).



Another twist is to construct an angle bisector for a pair of lines for which the vertex is off the page, and so extending the given lines is not possible (Leonard, 2014, §13, p. 58).



This might seem impossible, but in fact all we need to do is draw a line segment from *any* point on one line to *any* point on the other line (in purple below). Then, if we bisect both of the angles created (blue dotted lines), then the point where these angle bisectors intersect will lie on the angle bisector of the missing angle (because all three angle bisectors of a triangle are coincident). Now, we repeat this process for a different pair of points (or bisect the *supplementary* angles at the same pair of points – see the dashed lines below), and then join the two resulting points to give the angle bisector of the missing angle.



I think constructions constitute a really interesting and important topic in geometry, with huge potential for supporting students in thinking carefully and precisely about lines and angles. But, if we reduce it to *how-to* procedures that are actually more complicated (and most likely less accurate) ways of doing things students could easily do without them, then it is no wonder if they fail to see the value.

Notes

1. Of course, by the time a ruler has reached this level of usage, it is probably more accurate to call it a 'not-very-straight edge'.

2. It is tempting to suggest that these measurement methods are even more general. By measuring with a protractor, we can just as easily *trisect* any angle as bisect it, whereas, with the classical constructions, trisection of a general angle is impossible. Note, however, that using a ruler to trisect the *line segment* does not, of course, lead to trisection of the angle!

3. Overall, I think the least implausible way to set this up as a puzzle is to say that we want to draw these lines on a classroom whiteboard, and we don't have a board ruler or board protractor, but we can use any handy straight object (e.g., the edge of a textbook) and circular object (e.g., a paper plate with a small hole at the centre) to make straight lines and circles with.

4. I agree with Southall (2020) that, at least initially, drawing full circles, rather than just minimal arcs, helps students to appreciate better the role that the compasses are playing in providing us with *equal lengths*.

5. As Southall (2020) points out, there is no reason why the intersection points must be one above the line and one below, like this. By using two pairs of circles with different radii, the two intersections can be both on the same side of the line, allowing construction of a perpendicular bisector even when the given line is right on the edge of the paper.

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