Converting Recurring Decimals to Fractions

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This is a topic which I think is sometimes taught as an isolated technique within the curriculum. Perhaps converting recurring decimals to fractions is seen as the missing piece in the jigsaw of converting between fractions, decimals and percentages that students have learned at primary school. I think that the way this topic is approached depends on what the teacher thinks the point of it is. Modern calculators will do this conversion at the push of a button, so I think we are not teaching this topic to be 'useful'.

I thought about this when I heard some teachers discussing an assessment that one teacher had written for the entire department to use. Obviously I didn't record this exchange word for word, so this is my recollection of the essence of the disagreement, and I've used some artistic licence so as to try to make each side's position stronger!

The disagreement

Teacher A had included this question in a department assessment:

> Convert the recurring decimal $5.7\dot{3}$ into a fraction.
>
> Give your answer in its simplest form.
>
> Show your method.

Another teacher (Teacher B) commented that this was 'an easy one', because students are likely to know that $0.\dot{3}$ is $\frac{1}{3}$, but Teacher A didn't see why that was relevant. The method that Teacher A had expected the students to use was:

$$\begin{aligned} x &= 5.7|3333333\ldots && (1)\\ 10x &= 57.3|3333333\ldots && (2)\\ 9x &= 57.3 - 5.7 = 51.6 && (2) - (1)\\ x &= \frac{51.6}{9} = \frac{516}{90} = 5\frac{11}{15}. \end{aligned}$$

For Teacher A, $5.7\dot{3}$ was a *difficult* example, because $3 < 7$, meaning that the subtraction on line 3 of the above working involves an 'exchange'. Teacher A anticipated that the error, $57.3 - 5.7 = 52.4$, would lead to instances of the wrong answer,

$$x = \frac{52.4}{9} = 5\frac{37}{45}.$$

For this teacher, converting something like $3.5\dot{7}$ would have been easier, since this step would become a simple subtraction (with no exchanging):

$$\begin{aligned} x &= 3.5|7777777\ldots && (1)\\ 10x &= 35.7|7777777\ldots && (2)\\ 9x &= 35.7 - 3.5 = 32.2 && (2) - (1)\\ x &= \frac{32.2}{9} = \frac{322}{90} = 3\frac{26}{45}. \end{aligned}$$

Teacher B disagreed: Anything with recurring 3s in it should prompt them to think of thirds.

It turned out that Teacher B didn't teach the '$x = \cdots$, $10x = \cdots$ and subtract' method. Instead, she taught methods like this:

$$\begin{aligned} 5.73333333\ldots &= 5.7 + \left(\frac{1}{3} \div 10\right)\\ &= 5 + \frac{7}{10} + \frac{1}{30}\\ &= 5 + \frac{21+1}{30}\\ &= 5 + \frac{22}{30}\\ &= 5\tfrac{11}{15}. \end{aligned}$$

She argued that this kind of approach involved revision of more useful fraction techniques and took advantage of any 'easy' fractions buried within the starting decimal. For her, the point of this topic was to revisit ideas like that, rather than teach a 'new' method.

This was news to Teacher A, who now felt that Teacher B's class would gain an unfair advantage on his assessment, simply because he had happened to choose a recurring 3 in the question – which, based on *his* teaching approach, would be irrelevant. But Teacher B felt that her students would *always* gain an advantage with her method, whatever the numbers. For Teacher A's other example, they would just do:

$$\begin{aligned} 3.5777777\ldots &= 3 + \frac{1}{2} + \left(\frac{7}{9} \div 10\right)\\ &= 3 + \frac{1}{2} + \frac{7}{90}\\ &= 3\frac{52}{90}\\ &= 3\frac{26}{45}. \end{aligned}$$

Teacher A objected to this method because, although his students would know that $0.\dot{3} = \frac{1}{3}$, he *didn't* think they would know that $0.\dot{7} = \frac{7}{9}$, so they wouldn't be able to do it Teacher B's way.

Teacher B: If they don't know what happens with ninths, then they don't know the first thing about recurring decimals! We do lots of short division practice to see that $0.\dot{p} = \frac{p}{9}$, and $0.\dot{p}\dot{q} = \frac{pq}{99}$, etc., where p and q are any single digits. So we see how $\frac{1}{3}$ fits in as $\frac{3}{9} = 0.\dot{3}$, and even that $0.\dot{9}$ must be $\frac{9}{9} = 1$, although I admit that the students always argue about that one (Note 1)!

It turned out that Teacher A didn't know himself that something like $0.\dot{5}\dot{3}$ could instantly be converted to $\frac{53}{99}$, and indeed he wasn't sure that that would always work. Once he convinced himself that it would, he still regarded this as 'a trick', and not a valid method.

Teacher A: The question says 'Show your method', so students can't just write down the answer!

Teacher B: Well, your '$x = \cdots$, $10x = \cdots$ and subtract' so-called 'formal method' is no proof at all, because students at this level have no reason to think that they can multiply up and add and subtract infinite series like this, as though they were simply solving simultaneous equations! How do you convince them that doing that is valid?

Teacher A: They like that method – it's reducing the problem to algebra, which is a good habit. That way, every question is the same, whatever the numbers. If there's a n-digit repeating unit, you multiply by 10^n. Whereas with your method you have to think up a new way every time.

Teacher B: You might use your method for $5.7\dot{3}$, but would you use it for $5.\dot{3}$? Wouldn't you just write down $5\frac{1}{3}$? Or, if the question was $5.\dot{7}$, wouldn't you just write down $5\frac{7}{9}$?

Teacher A: I would, but those are easy ones. I wouldn't give them easy ones like that in an assessment.

Teacher B: All questions like this are either 'easy ones' or just a step or two away from being 'easy ones'. Is $5.\dot{2}\dot{1}$ an 'easy one' or not? Whatever you have, you just fiddle around with it until it *is* easy: reducing the problem to a simpler case – another good mathematical habit! It works just as well even for those calculator-proof questions, where digits are represented by letters.

Teacher A: Hang on! You have to use my method at some point to prove that $0.\dot{p} = \frac{p}{9}$, $0.\dot{p}\dot{q} = \frac{pq}{99}$, and so on, surely?

Teacher B: No I don't. I just do them by short division. You can prove $\frac{p}{9} = 0.\dot{p}$ by exhaustion, by doing all 9 possibilities. But you don't even really need to do that. Once you establish that $\frac{1}{9} = 0.\dot{1}$, then it follows that $\frac{p}{9}$ is p lots of this, so each digit 1 gets multiplied to make a digit p, and therefore $\frac{p}{9} = 0.\dot{p}$.

Alternatively, all you need to do is one 'generic example' of a division, say $\frac{7}{9}$, while looking carefully at what happens.

$$\begin{array}{r} 0. \\ 9\,\overline{)\,7.0\,0\,0\,0\,0\,0\,0\,0\,0} \end{array}$$

So we ask:

How many 9s go into 70?

The answer is 7, with remainder 7. And our remainder 7 makes us another 70 for when we move to the next digit.

If the dividend 7 were some other number, like 4, say, we would ask:

How many 9s go into 40?

The answer is 4, with remainder 4.

Why are the answer and the remainder always both equal to the number we are dividing? It's because $9 \times 7 + 7 = 10 \times 7$,
or $9 \times 4 + 4 = 10 \times 4$,
or, in general, $9n + n = 10n$.

So it's clear that this will always happen whatever number we divide by 9.

Similarly, if we think about, say $\frac{53}{99}$:

$$\begin{array}{r} 0. \\ 99\,\overline{)\,5\,3.0\,0\,0\,0\,0\,0\,0\,0\,0} \end{array}$$

then this time we have to ask:

How many 99s go into 5300?

The answer is 53, with remainder 53.

(Teacher A is amazed by this!)

It's just because $99 \times 53 + 53 = 100 \times 53$. So, our quotient has to be just a 'zero point' followed by repeated 53s forever (Note 2)!

Conclusion

I wonder what you make of the positions that these two teachers adopt. It is easy to dismiss Teacher A as 'the baddy' and Teacher B as 'the goody'. Teacher A has a fixed method that he wants to cling to in almost every case, whereas Teacher B is more open-minded and adaptable – flexing to whatever is going to work best for the situation. But I think it isn't as simple as that. Teacher A wants to be *rigorous* and to teach a powerful algebraic method that will give his students certainty about how to solve any problem. But Teacher B doesn't think Teacher A's method is as rigorous as it appears, and wants to leverage recurring decimals as an opportunity to review equivalent fractions and addition and subtraction of fractions. Perhaps it comes down to why we teach this topic in the first place. Is it a 'handy technique' for the toolbox or a means to an end as an opportunity to deepen students' knowledge and understanding of fractions?

Notes

1. *How many 9s are in 90?* We could say "10", or we could say "9, remainder 9", which produces another 90. Now, "How many 9s are in 90?"...
2. One way to see how well students understand what is going on is to change to, say, base 7 (Foster, 2022). Now, for example, $\frac{5}{6} = 0.\dot{5}$, and through exactly analogous reasoning. This free online calculator is fun for exploring such things: https://planetcalc.com/862/

Reference

Foster, C. (2022, October 27). Butterfly effects when adapting tasks. [Blog post]. https://blog.foster77.co.uk/2022/10/butterfly-effects-when-adapting-tasks.html

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Twelve Intersecting Holes and a Cube

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Spatial visualization is defined as the ability or skill drawn upon to mentally transform or manipulate spatial properties of an object (Lowrie, Logan, & Hegarty, 2019). It also involves "seeing" things that are not in front of you. The connection between success in mathematics and the ability to visualize in space cannot be overemphasized. Moreover, spatial visualization is a skill that can be honed through providing students with rich mathematical tasks or problems that create a context for spatial visualization. However, traditional textbooks don't offer a plethora of rich activities that create contexts for students to sharpen their spatial visualization skills. Most of them don't go much beyond isometric drawings of solids made of cubes. In this short article we will look at a problem that can be explored virtually on a computer or mobile.

The Problem

Suppose we started with an $8 \times 8 \times 8$ cube made using 512 unit cubes. Four holes were then drilled through each pair of opposite faces as shown. That is, there were altogether 12 holes. The area of the cross-section of each hole is three units. Each hole is L-shaped as shown in Figure 1. The six external faces of the resulting solid are congruent to one another and each hole is perpendicular to the faces it passes through. Determine the volume of the resulting solid Figure 1.

To explore the solid virtually go to

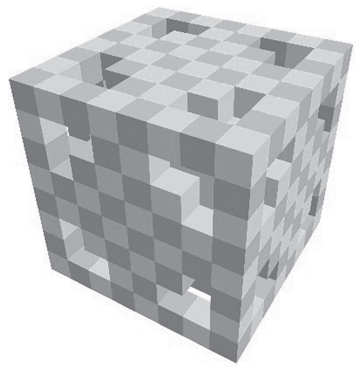

Figure 1

www.3dslash.net/slash.php?partner=viewer&content=4cd05f66e87c2719df57b25aa3bf208e37b51263295f71046b96ba35fc8bdd0b&autoplay=1&rulers=0&zoom=0 Alternatively, you can use the QR code to access the solid virtually.

When you are exploring the solid on a computer screen you can drag the solid by right-clicking and dragging your mouse. The solid can be turned around by left-clicking and dragging the mouse.

The solid used in this problem was made using 3D Slash (Huet & Jacomet, 2014). This software