Identity Crisis

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How often do you use the identity symbol ≡? When do you use it, and how consistent are you about this?

The identity sign is supposed to function as a 'super equals sign', meaning that, not only do two expressions *happen* to be equal sometimes, but they are *always* equal, under *all* circumstances. So, for example, $2x$ *could be* equal to 10, if x were equal to 5, and so we could write $2x = 10$ as an *equation* to be solved [Note 1]. But, $2x$ is *always* equal to $x + x$, regardless of the value of x, and so we might write $2x \equiv x + x$ as an *identity*, which is, in a sense, *more than an equation*. Using the 'always, sometimes, never' classification for mathematical statements (Swan, 2002), we have three different symbols, which, on the domain of the integers, could give us:

Always equal:	$2x \equiv x + x$
Sometimes equal:	$2x = 10$
Never equal:	$2x \neq 11$

I think I tend to make a point of using ≡ when working with trigonometric identities and equations – no doubt the word 'identity' in the topic title cues me into this. I like to distinguish an *identity* like $\cos^2x + \sin^2x \equiv 1$, which is true for all real x, from a similar-looking *equation,* like $\cos^2x - \sin^2x = 1$, which is true only for certain values of x; in this case, $x = n\pi$, $n \in \mathbb{Z}$. This feels like an important distinction, and the ≡ symbol helps to make it. The identity symbol is also sometimes useful if you want to distinguish, say, $f(x) \equiv 0$, the zero function (zero everywhere), from writing $f(x) = 0$, where you are solving to find the root(s) of a function f that is *not* the zero function.

However, I find the identity symbol a bit problematic because it's hard to be really consistent about when it ought to be used. The identity $2x \equiv x + x$ is unproblematic, because it's true for all x whatsoever, but what about equations that are *almost* identities, such as:

$$\frac{1}{x}+\frac{1}{x}=\frac{2}{x}$$

which is always true, except if $x = 0$. Do they merit the identity symbol? It feels a bit harsh to deny this the identity symbol, just because of one exceptional value, for which the entire thing would be undefined anyway. But then we often tell students that one counterexample is enough to disprove something, so it would seem a bit strange to make an exception here, unless it were somehow clear that zero was not in the domain. If you are going to be strict about this policy, then $(a - b)(a + b) \equiv a^2 - b^2$ should be written as an identity but

$$a+b=\frac{a^2-b^2}{a-b}$$

should not, because it is invalid when $a = b$. But I am not sure that this level of pedantry helps anyone. Maybe we should just use the identity symbol whenever the equation is true for all values in the 'natural' domain (i.e., the obvious, relevant, common-sense domain).

This would still be problematic for something like the binomial expansion:

$$(1+x)^n = 1+\frac{nx}{1!}+\frac{n(n-1)x^2}{2!}+\dots$$

Is this an identity or not? If n is a non-negative integer, then it's true for all x, so we could use the identity symbol and write

$$(1+x)^n \equiv 1+\frac{nx}{1!}+\frac{n(n-1)x^2}{2!}+\dots,\ n\in\mathbb{Z}^+$$

But, for other values of n, the expansion is valid only for $|x| < 1$, so then perhaps it *wouldn't* class as an identity, even though it is still true for infinitely many values of x. This seems to raise the issue of what "for all values" means in any particular context, and there is an imprecision about the identity symbol if we are not specific about the domain, which in school mathematics we often aren't [Note 2]. The statement

$$\sqrt{a}\sqrt{b}=\sqrt{ab}$$

is an identity, but not if either a or b is negative. The 'rule of indices' that $x^a \div x^b = x^{a-b}$ is an identity, so long as $x \neq 0$, and so on.

If the identity symbol should be used whenever something is 'always' true, then what about arithmetic? For example, it is 'always' true that $1 + 1 = 2$, so does this mean that we should be writing all of arithmetic with identity symbols, as $1 + 1 \equiv 2$? It is not clear what would be gained by doing this, but I am left feeling that if I cannot use this symbol consistently then perhaps it would be better to avoid it altogether. After all, we get by well enough using *inequalities* without making this distinction. The arithmetic-mean-geometric-mean (AM-GM) inequality

$$\sqrt{ab}\le\frac{a+b}{2}$$

is true for all non-negative a, b, so it kind of has the status of an *inequality-identity*. But, as far as I know, we have no symbolic way of distinguishing this from an inequality like

$$ab\le\frac{a+b}{2}$$

which can be *solved* to show that it holds if and only if

$$a \le \frac{b}{2b-1}.$$

Does our inability to mark the former as an identity cause any problems? I am not sure that it does.

When I watch mathematics lessons, I very rarely see the identity symbol used on the board. Perhaps we should be using ≡ every time we write something that is true for all x, such as in common 'simplifying' situations, like:

$$\begin{aligned}(x+5)(x-3)+2(2x-3)+5 \\ &\equiv x^2+2x-15+4x-6+5 \\ &\equiv x^2+6x-16 \\ &\equiv (x+8)(x-2).\end{aligned}$$

I am left thinking that either we should be using the identity symbol far more often than we do, whenever we write an equation that is true for the entirety of some relevant or implied domain, or else we should perhaps not bother with it, and declutter our symbol palette.

Postscript

I am grateful to one of the Editors for drawing my attention to his own practice of restricting the word *inequality* to always-true statements, such as $3 < 5$, and using the word *inequation* for algebraic situations in which there is an unknown to be evaluated, such as $3 + x < 5$. This would enable a distinction to be made between

$$\sqrt{ab} \le \frac{a+b}{2} \text{ and } ab \le \frac{a+b}{2}$$

by calling the former an inequality and the latter an inequation. In a similar way, a statement such as $3 + 2 = 5$ would be termed an *equality*, in contrast to an *equation*, like $3 + x = 5$. I would be very interested to hear readers' thoughts on the value of this practice or others.

Note

1. Sometimes called a *conditional* equation.
2. The description 'always' is 'always' problematic, because it 'always' just means 'in the context in which we are operating'. And, surely, 'true' is an absolute, so shouldn't really be qualified!

Reference

Swan, M. (2002). 'Always, sometimes or never true?', *Mathematics Teaching*, **181**, 32–33.

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