Logarithms and their Bases

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Which is the larger of these two expressions

$$\frac{\log_7 6}{\log_7 5} \text{ or } \frac{\log_3 6}{\log_3 5} \text{ ?}$$

I will leave that question hanging while I explain why I ask it.

I think that students often have a good understanding of many of the laws of logarithms, by linking them to the corresponding laws of indices (see Note). For example, they know that $b^p b^q \equiv b^{p+q}$, and so

$$\log_b(b^p b^q) \equiv \log_b(b^{p+q}) \equiv p + q.$$

Re-expressing this in terms of $m = b^p$ and $n = b^q$ gives us the identity $\log_b mn \equiv \log_b m + \log_b n$. For analogous reasons, division involves subtracting the logarithms and powers involve multiplication, and these can be fairly easy to remember, because they fit a nice pattern.

However, even for myself, I find the *change of base formula* much less intuitive. For two bases, b and c, this is often written as:

$$\log_b a = \log_c a \, . \log_b c.$$

The chain of 'a to b' being 'a to c times c to b' may make it easy for the student to remember (going from a to b via c). But I think that this formula is rarely well understood. It is easy to get lost in the proof amid all the different symbols. If $x = \log_c a$ and $y = \log_b c$, then in exponential form we have $a = c^x$ and $c = b^y$, so that $a = (b^y)^x = b^{xy}$. This means that

$$\log_b a = xy = \log_c a \, . \log_b c.$$

So, it's true, but this dance with the symbols doesn't really give me much insight, because I find myself getting bogged down in all of the different letters, and struggling to remember which letters are which.

However, I think there is a much easier way to think about this. Suppose that we want to express $\log_7 6$ in terms of logarithms to base 3. We can write

$$\log_7 6 = \log_7\left(3^{\log_3 6}\right).$$

This looks complicated, but it is really just writing the number 6 in a funny way, as a power of 3. Which power of 3 will it be? By definition, it will have to be the $(\log_3 6)^{th}$ power of 3, if we want it to come out to 6. So, $6 = 3^{\log_3 6}$. With familiarity, this becomes a useful 'trick' when working with logarithms, and is something that I think is worth practising. Students may think of the '3 to the power of' as the inverse of 'log to base 3', and so they 'cancel out' when they are applied successively to 6, in a similar way to how $\log_3(3^6)$ is also equal to 6. But I prefer just to think that the power to which we need to raise 3 to get 6 is exactly what we define as the logarithm of 6 to base 3, so it's kind of a tautology to say that $6 = 3^{\log_3 6}$.

This is the only tricky part. Once we've written 6 in this awkward way, we can just take logarithms on both sides and use the power rule that $\log_b(a^n) = n \log_b a$ to get

$$\log_7 6 = \log_3 6 . \log_7 3,$$

which is our change-of-base rule.

I find this much easier to follow than the version with letters, even though of course it's exactly the same. Indeed, you can say that the numerical version is just as general: the 3, 6 and 7 are just symbols. We didn't use any facts about those particular numbers, such as $3 \times 2 = 6$ or $6 + 1 = 7$, so what we've done must work for 'all values of 3, 6 and 7'! We could forget that 3, 6 and 7 are numerals at all, or we could perhaps write them in a squiggly way (3, 6, 7) and say that they aren't actually numbers, but just 'symbols' that happen to look a bit like numerals. And, if we wish, we can of course do the same thing more formally with a, b and c.

We wish to replace the question mark below with a power of c:

$$\log_b a = \log_b ?$$

So, we write a in an awkward way, as $c^{(\log_c a)}$:

$$\log_b a = \log_b\left(c^{(\log_c a)}\right).$$

Then, it's just one step to obtain:

$$\log_b a = \log_c a \, . \log_b c.$$

I certainly prefer this proof to the one above that unnecessarily introduces x and y and gives me too many letters to easily keep track of.

Quotients

Does this help me see why the two quotients with which we began are equal? In general, a ratio of logarithms to the same base *doesn't depend on the base*. We can see this if we begin with

$$\log_b a = \log_c a \, . \log_b c$$

and divide both sides by $\log_b c$, to obtain

$$\frac{\log_b a}{\log_b c} = \log_c a,$$

in which both logarithms on the left-hand side have the same base. Here, the b is arbitrary, since it doesn't appear on the right-hand side, so

$$\frac{\log_{b_1} a}{\log_{b_1} c} = \frac{\log_{b_2} a}{\log_{b_2} c} = \frac{\log_{b_3} a}{\log_{b_3} c} = \cdots = \log_c a.$$

This means that both

$$\frac{\log_7 6}{\log_7 5} \text{ and } \frac{\log_3 6}{\log_3 5}$$

are equal to $\log_5 6$, and so must be equal to each other. Perhaps it's better to remember the change of base rule in quotient rather than product form, in words (Foster, 2022), as 'ratios of logarithms to the same base are independent of the base'? Incidentally, I find that students rarely connect the word 'base' in logarithms with the same word in the context of writing numbers in different bases. The logarithm of 1000 to base 10 is 3 because the 1 comes in the 10^3 column. So, the number that is written as 1000 in base 7, say, will also have a logarithm *to base 7* of 3, because the 1 comes in the 7^3 column when $7^3 = 343$ is written in base 7.

Differentiating powers

The 'trick' of using logarithms to express any number as a power of any other number is often convenient. For example, if you want to differentiate $y = a^x$, then you can take logarithms of both sides and differentiate implicitly:

$$\begin{aligned} y &= a^x \\ \ln y &= \ln(a^x) \\ \ln y &= x \ln a \\ \frac{1}{y}\frac{dy}{dx} &= \ln a \\ \frac{dy}{dx} &= y \ln a = a^x \ln a. \end{aligned}$$

But this seems like quite a lot of steps, and substituting back in for y at the end feels as though I lost contact with what I was doing. It also requires implicit differentiation, which students might not yet know.

Compare that with saying that $y = a^x$ is just an exponential function expressed in terms of the 'wrong' base. If only it were base e, we could differentiate it easily. So, let's *make* it base e, by writing $y = e^?$. Since, by definition of what ln means, $a = e^{\ln a}$, we have

$$y = a^x = \left(e^{\ln a}\right)^x = e^{x \ln a}.$$

Differentiating this is no harder than differentiating something like $y = e^{3x}$, since $\ln a$ is just a number, like 3. The ln part of $\ln a$ makes it look like 'a function', but since a is a constant, $\ln a$ is just as much a constant as $\sqrt{a}$ or a^3. So,

$$\frac{dy}{dx} = (\ln a)e^{x \ln a} = (\ln a)a^x.$$

Not only is this, I think, much quicker and easier, but it is helpful for seeing that a^x is just a scaling of e^x. When $a = e$, the factor of $\ln a$ reduces to 1, revealing e^x as the *special* exponential function that is exactly its own derivative. All of the other exponential functions have derivatives that are multiples of themselves, but for e^x the multiplier is 1. To me, this is memorable and gives more insight into what is going on.

Note

I am not sure why we call these 'rules' or 'laws', whereas in trigonometry we call things like $\sin^2 x + \cos^2 x \equiv 1$ 'identities'. I would rather say 'the logarithmic identities', but I don't, because no one else seems to.

Reference

Foster, C. (2022). 'Are words sometimes better than formulae?', *Mathematics in School*, 51(5), pp. 12–14.

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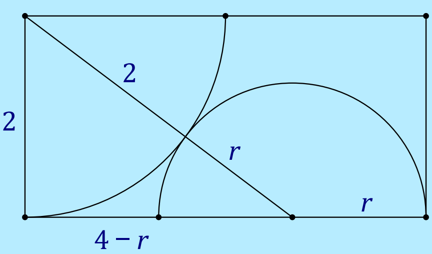
Solution to the First Domino Problem



The radius of the quadrant is 2.

Let the radius of the semicircle be r. Then the base is divided into line segments of length r and $4 - r$. If a line joins the centres of the quadrant and semicircle, it has length $r + 2$ and this is also the hypotenuse of a right-angled triangle.

By Pythagoras' Theorem, $(r + 2)^2 = (4 - r)^2 + 2^2$.

This leads directly, to $r = \frac{4}{3}$, the radius of the semicircle.

Note that the triangle is (3, 4, 5).