Playing with asymptotes

Colin Foster

I always enjoy teaching curve sketching because of the piece-by-piece detective work involved in putting various bits of information together until there remains only one possibility for what the curve must be like. I particularly enjoy thinking about asymptotes, because there are many counterintuitive and surprising features to explore with students.

When sketching rational functions of the form

$$y = \frac{f(x)}{g(x)},$$

where both $f(x)$ and $g(x)$ are polynomials, students learn to find vertical asymptotes by setting the denominator $g(x)$ to zero and solving the resulting equation. For example, to find the vertical asymptotes of

$$y = \frac{x-2}{x^2-1},$$

the student would write $x^2 - 1 = 0$ and solve this to find $x = -1$ and $x = 1$ as the two vertical asymptotes (Figure 1). When doing this, students are sometimes so focused on the denominator that they fail to notice if the function they are presented with is not in its simplest form. For example, if asked to find the vertical asymptotes of

$$y = \frac{x-1}{x^2-1},$$

they might draw the same conclusion (the vertical asymptotes are $x = -1$ and $x = 1$) and be surprised when sketching the curve to discover that the $x = 1$ asymptote does not seem to exist (Figure 2). This happens because

$$\frac{x-1}{x^2-1} = \frac{x-1}{(x-1)(x+1)} = \frac{1}{x+1},$$

so there is no $x = 1$ asymptote. Technically, although

$$\frac{x-1}{x^2-1} = \frac{1}{x+1},$$

the curve

$$y = \frac{x-1}{x^2-1}$$

has the point (1, 0) missing, since $x = 1$ is not in its domain, whereas

$$y = \frac{1}{x+1}$$

has no such hole. It is interesting to explore the family of curves

$$y = \frac{x-a}{x^2-1},$$

where a has values such as 0.9 or 0.99, close to 1.

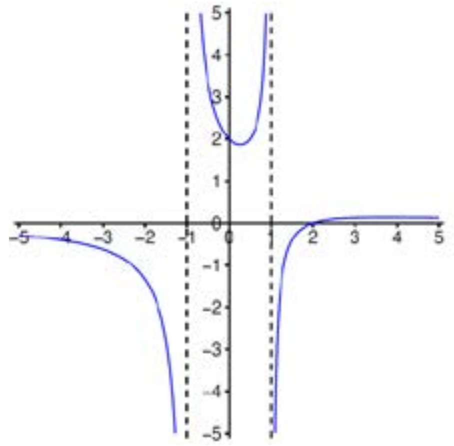


Figure 1. The graph of $y = \frac{x-2}{x^2-1}$.

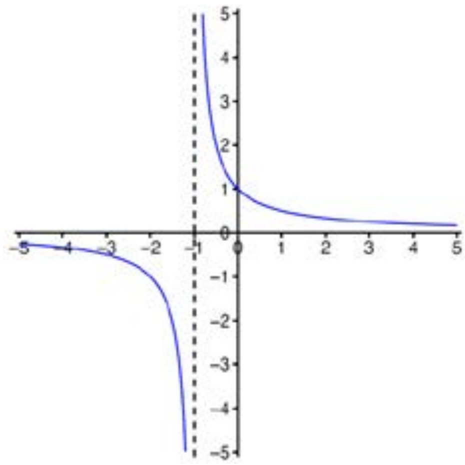


Figure 2. The graph of $y = \frac{1}{x+1}$.

Students are also often surprised, when asked to sketch the graph of the similar-looking equation

$$y = \frac{x-2}{x^2+1}$$

with just a single sign swap in the denominator, that the curve looks completely different (Figure 3, next page). To find vertical asymptotes, they write $x^2 + 1 = 0$, and they are supposed to conclude that there are no real solutions to this equation, and therefore no vertical asymptotes. However, one student at this point wrote "No *real* vertical asymptotes", suggesting that perhaps there could be 'complex' ones. If students are simultaneously just beginning to learn about complex numbers, they may even conclude that there are asymptotes at $x = \pm i$. One student interpreted that in relation to the Argand diagram by concluding that this meant that there were *horizontal* asymptotes at $y = \pm 1$. It is understandable that students think that now that they know about complex numbers they need no longer be thwarted when faced with an equation like $x^2 + 1 = 0$ and ought to be able to solve it.

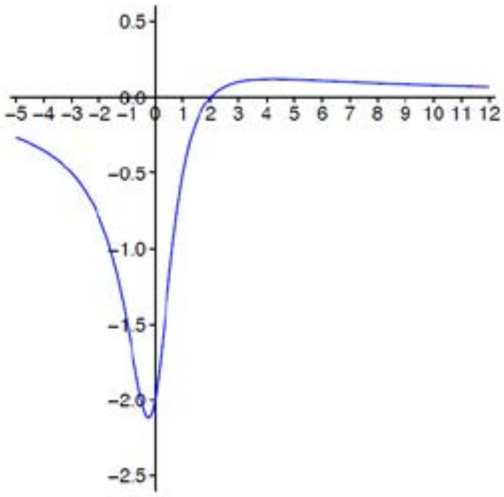


Figure 3. The graph of $y = \dfrac{x-2}{x^2+1}$.

Horizontal asymptotes

If vertical asymptotes come from setting the denominator equal to zero, students may feel that, 'by symmetry', horizontal asymptotes must come from setting the numerator to zero. Mathematics is supposed to be all about patterns. But there are wrong patterns as well as right ones (Foster, 2020)! Setting the numerator to zero finds us the zeroes of the function, not the asymptotes.

It is usual to introduce horizontal asymptotes by saying that they are all about what happens when the magnitude of x gets very large, both positively and negatively. It is nice to see this by zooming out on the x-axis using graphing software, leaving the y-axis unchanged. Any little wiggles or intersections near the y-axis (i.e. for small-magnitude values of x) become hard to see, and a function like

$$y = \frac{x-2}{x^2-1}$$

ends up looking like the more familiar $y = 1/x$ for large $|x|$ (Figure 4).

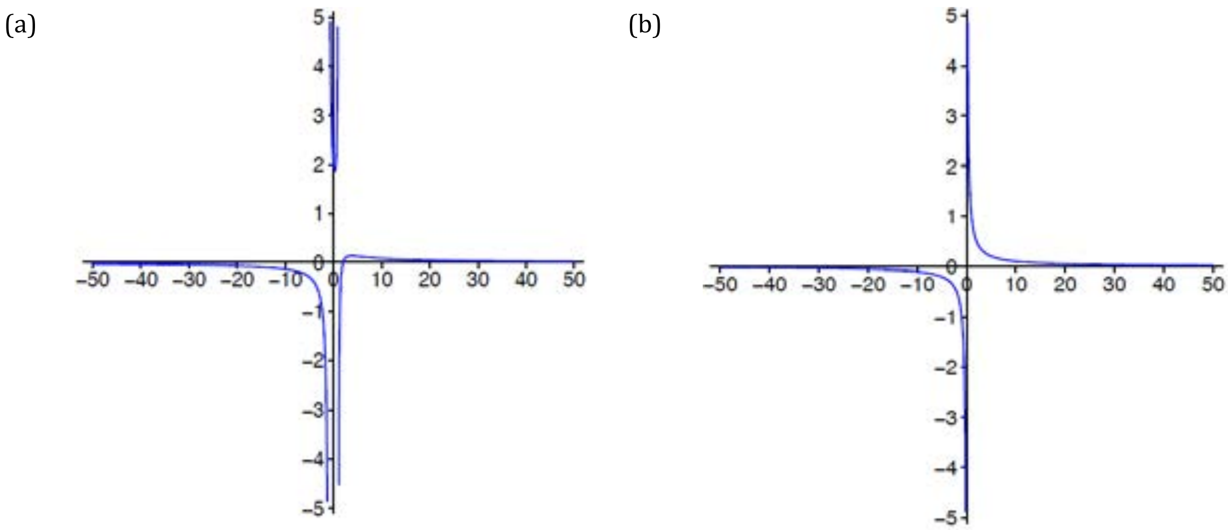


Figure 4. The graph of (a) $y = \dfrac{x-2}{x^2-1}$ approaches (b) $y = \dfrac{1}{x}$ for large $|x|$.

Having seen this graphically first, we can then see it algebraically. Informally, we might say that subtracting 1 or 2 from x^2 or x is going to make next to no difference, once $|x|$ becomes very large. (A million minus 2, divided by a trillion minus 1, is essentially a million divided by a trillion, which is essentially zero).

So, $$y = \frac{x-2}{x^2-1} \to \frac{x}{x^2} = \frac{1}{x}.$$

Therefore, since $y = 1/x$ has the x-axis as a horizontal asymptote, so will

$$y = \frac{x-2}{x^2-1}.$$

More formally, we can divide both the numerator and the denominator by x^2:

$$y = \frac{x-2}{x^2-1} = \frac{\frac{1}{x} - \frac{2}{x^2}}{1 - \frac{1}{x^2}} \to \frac{0-0}{1-0} = 0,$$

giving a horizontal asymptote of $y = 0$.

It is useful to ask students how they would modify the function

$$y = \frac{x-2}{x^2-1}$$

to obtain a horizontal asymptote that is some other horizontal line, that isn't the x-axis. So often, horizontal asymptotes are the x-axis (e.g. exponential functions) that students may assume that this is the only possibility. For example, suppose that they wanted this process to lead to $y = 3$ as the horizontal asymptote. Working backwards, we will need something like

$$\frac{\frac{1}{x} - \frac{2}{x^2} + 3}{1 - \frac{1}{x^2}},$$

which would come from

$$y = \frac{3x^2 + x - 2}{x^2 - 1}.$$

This simplifies to

$$y = \frac{(3x - 2)(x + 1)}{(x - 1)(x + 1)} = \frac{3x - 2}{x - 1} \text{ (Figure 5).}$$

We can see that the graph of this function has a horizontal asymptote at $y = 3$, by writing

$$\frac{3x - 2}{x - 1} \to \frac{3x}{x} = 3,$$

or by dividing the numerator and denominator of

$$\frac{3x - 2}{x - 1}$$

by x, as above. We can also see it by 'dividing out' the denominator from the numerator:

$$\frac{3x - 2}{x - 1} = \frac{3(x - 1) + 1}{x - 1} = 3 + \frac{1}{x - 1}.$$

Students often find this a tricky procedure, but it is exactly analogous to converting an improper fraction to a mixed number:

$$\frac{22}{7} = \frac{3 \times 7}{7} + \frac{1}{7} = 3\frac{1}{7}.$$

This corresponds to substituting $x = 8$ into the expression above. And the expression

$$3 + \frac{1}{x - 1}$$

clearly tends to 3 as $|x|$ goes to infinity.

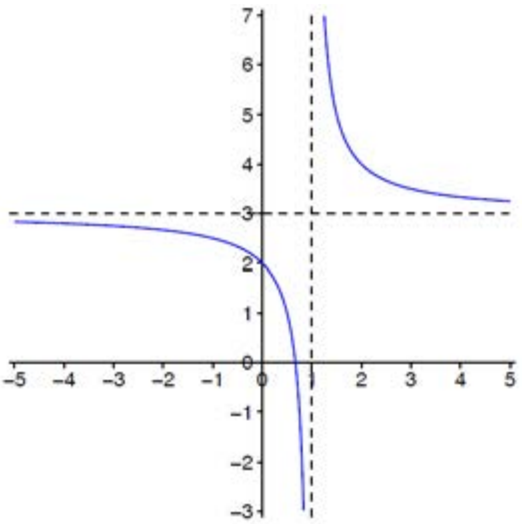


Figure 5. The graph of $y = \frac{3x - 2}{x - 1}$.

A final way to discover this horizontal asymptote is to rearrange the equation. Unlike the original function that contained an x^2, we can make x the subject of

$$y = \frac{3x - 2}{x - 1}.$$

Multiplying up, we have

$$xy - y = 3x - 2$$
$$x(y - 3) = y - 2,$$

giving

$$x = \frac{y - 2}{y - 3}.$$

Although this feels like finding the inverse function, it is not quite the same, because here it is not necessary for x to be a function of y in order to write this relation.

Now we can see from the denominator that $y = 3$ corresponds to $|x| \to \infty$, just as before. I think it is really helpful for students to encounter all of these equivalent ways of seeing the same thing.

Relating horizontal asymptotes to vertical asymptotes

Students may feel that these arguments about what happens when $|x| \to \infty$ for horizontal asymptotes seem entirely different from checking when the denominator is zero for vertical asymptotes. But, actually, they are very closely related. Looking for a zero denominator for an expression equal to y is exactly equivalent to checking for values of x for which $|y| \to \infty$. It is not quite symmetrical, however, as we have seen, since y must be a single-valued function of x, but x doesn't have to be a single-valued function of y.

Because of this, a function can have as many vertical asymptotes as it likes; the function $y = \tan x$, for example, has infinitely many. But the number of possible horizontal asymptotes is restricted by the fact that each x value must map onto a single y value. Since there are only two possible infinite x limits ($+\infty$ and $-\infty$), there is a maximum of two possible y values that could be associated with them. This means that a function can have zero, one or two horizontal asymptotes, but no more than that.

For example, $y = 3x - 1$ (or any polynomial of degree 1 or more) has no horizontal asymptotes, a function like

$$y = \frac{3}{x - 1}$$

has one horizontal asymptote (at $y = 0$), and to find a function with two horizontal asymptotes we need the function to behave differently for large positive x and large negative x. Students could be invited to try to come up with an example of a function with two horizontal asymptotes. One way to do this is to 'cheat' by using a piecewise function. For example, we could use the right-hand portion of

$$y = \frac{3x - 2}{x - 1},$$

which, as we have seen, has a horizontal asymptote at $y = 3$, and the left-hand portion of

$$y = \frac{3}{x - 1},$$

which has a horizontal asymptote at $y = 0$ (Figure 6).

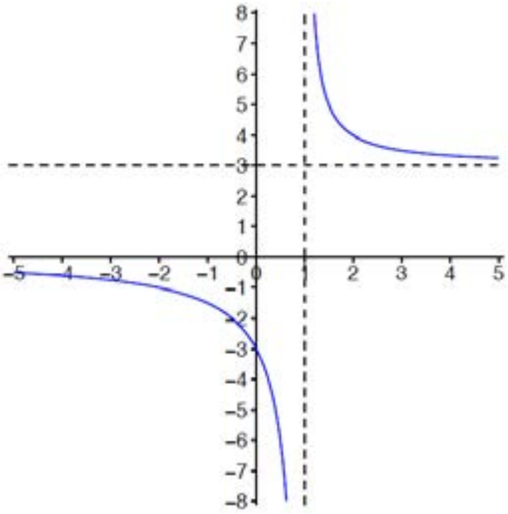


Figure 6. The graph of $y = \begin{cases} \dfrac{3}{x-1} & \text{if } x < 1 \\ \dfrac{3x-2}{x-1} & \text{if } x > 1 \end{cases}$

A single rational function (i.e. one with numerator and denominator both polynomials) never has more than one horizontal asymptote, but other functions do. For example, sigmoid functions, such as $y = \tan^{-1} x$, do, as does

$$y = \frac{x}{\sqrt{1+x^2}} \text{ or } y = \frac{x}{1+|x|} \quad \text{(Figure 7, to the right).}$$

A step function could also be a (perhaps trivial) example.

Another difference between vertical and horizontal asymptotes is that a smooth curve can't cross a vertical asymptote, since its value becomes arbitrarily large as it gets close to the x value of the asymptote. But there is no prohibition against a curve crossing a horizontal asymptote, and indeed it may do so as often as it wishes – even infinitely often; for example,

$$y = \frac{\sin x}{x}.$$

Horizontal asymptotes are about what a curve does far out to the left and right, for large $|x|$. What the curve does nearer to the y-axis is of no relevance to this. This can be a problem for students if their informal definition of an asymptote is 'a line that the curve gets closer to but never touches'. Figures 1 and 3 show curves crossing the horizontal asymptote $y = 0$ before eventually getting arbitrarily close to it, as $|x|$ increases.

Oblique asymptotes

It can be a surprise to students that vertical asymptotes and horizontal asymptotes are not the only two possibilities. There is no reason why asymptotes must be parallel to the coordinate axes. In a rational function, if the degree of the numerator is 1 more than that of the denominator, then we get oblique (or slant) asymptotes.

(a)

(b)

(c)

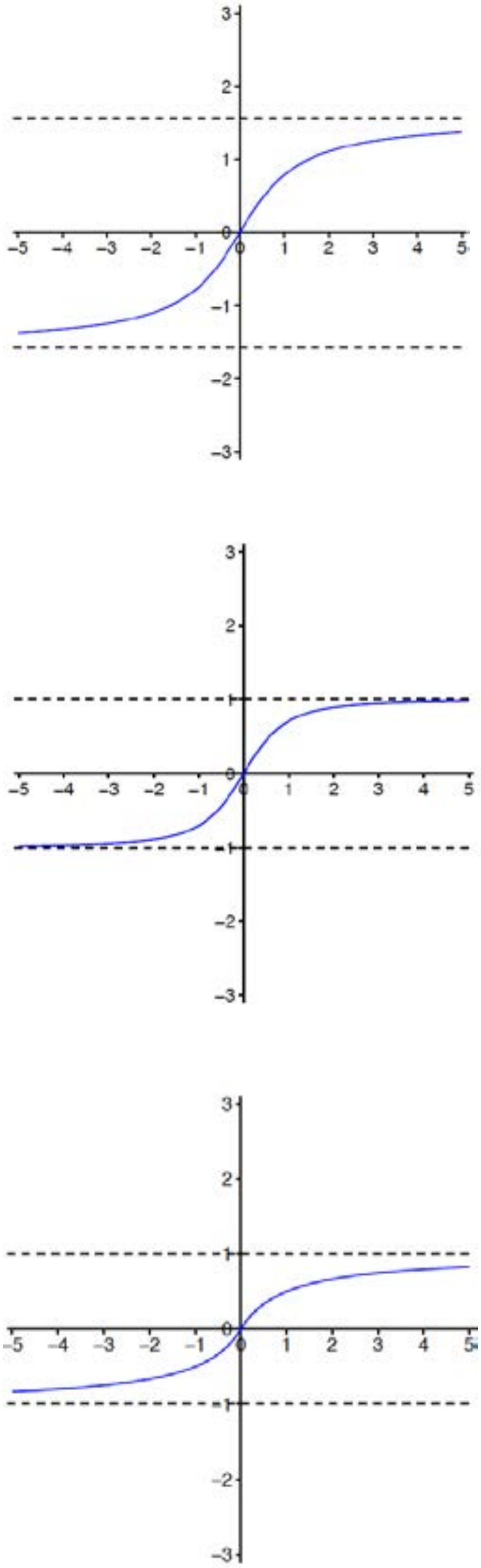


Figure 7. The sigmoid graphs of (a) $y = \tan^{-1} x$, (b) $y = \dfrac{x}{\sqrt{1+x^2}}$ and (c) $y = \dfrac{x}{1+|x|}$.

Suppose we want to find a function with an oblique asymptote of $y = x + 2$. This means that the function must approach $x + 2$ as $|x|$ increases, so it must be of the form

$$y = x + 2 + (\text{parts that } \to 0 \text{ as } x \to \infty).$$

We can easily make such a function; for example,

$$y = x + 2 + \frac{1}{x-1}.$$

Now, to disguise it a little, we can put this over a common denominator:

$$\begin{aligned} y &= x + 2 + \frac{1}{x-1} \\ &= \frac{(x+2)(x-1)+1}{x-1} \\ &= \frac{x^2+x-1}{x-1}. \end{aligned}$$

If we plot this, and zoom out, we can clearly see that it looks just like $y = x + 2$ for large $|x|$ (Figure 8). The curve doesn't level off horizontally, like curves with horizontal asymptotes do; it gets arbitrarily close to a diagonal line with gradient 1 and intercept 2.

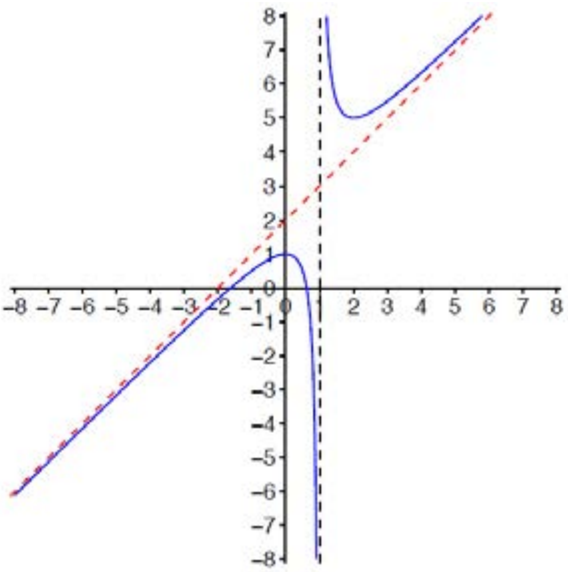


Figure 8. The graph of $y = \frac{x^2+x-1}{x-1}$ (blue) when zoomed out looks like the graph of $y = x + 2$ (red).

But how could we discover that the oblique asymptote would be $y = x + 2$ if we hadn't deliberately constructed it that way ourselves? We would just have to divide out

$$y = \frac{x^2+x-1}{x-1}$$

and write it in a more transparent form.

We can write:

$$x^2 + x - 1 = (x-1)(\ldots\ldots\ldots),$$

where the dots inside the brackets indicate as-yet-undetermined terms. Since we want a quadratic answer, this second bracket must also be linear, so there will be just two terms, a term in x and a constant.

The first part of the empty bracket must be x, in order to multiply the other x to make the required x^2:

$$x^2 + x - 1 = (x-1)(x \ldots\ldots).$$

The byproduct of placing this x there is that it can't help being forced to multiply the -1 as well, to produce $-x$, whereas we actually want $+x$. We can fix that by putting $+2$ as the second term in the bracket:

$$x^2 + x - 1 = (x-1)(x+2) \ldots$$

Now, we have created $x^2 + x - 2$, whereas we wanted $x^2 + x - 1$. The first two terms are correct, but the constant is wrong, so we need to add 1 to correct it, giving:

$$x^2 + x - 1 = (x-1)(x+2) + 1.$$

This process of division 'by inspection' is reminiscent of the process of completing the square.

Our division tells us that

$$\begin{aligned} y &= \frac{x^2+x-1}{x-1} \\ &= \frac{(x-1)(x+2)+1}{x-1} \\ &= (x+2) + \frac{1}{x-1}, \end{aligned}$$

as we had originally. Doing the division and discarding the fractional part, which always goes to zero as $|x|$ goes to infinity, will always give us the oblique asymptote.

It is worth students being aware that there are wrong approaches to finding oblique asymptotes. Someone might plausibly say that for large $|x|$, we can ignore the –1s in

$$\frac{x^2+x-1}{x-1}$$

to obtain

$$\frac{x^2+x}{x} = \frac{x+1}{1},$$

and conclude (wrongly) that the oblique asymptote is $y = x + 1$, rather than the correct line, $y = x + 2$. We can clearly see from Figure 8 that $y = x + 1$ is not asymptotic to this curve.

We get the same wrong answer, perhaps more plausibly, by first dividing the numerator and denominator by x, giving

$$\frac{x + 1 - \frac{1}{x}}{1 - \frac{1}{x}}$$

which also appears to tend to $x + 1$ as $|x|$ tends to infinity. How do these seemingly innocent steps manage to give us the wrong oblique asymptote?

The problem is that when we reason about asymptotic behaviour by considering large $|x|$, we have to throw away everything except the most significant term in each expression. In

$$\frac{x^2+x-1}{x-1},$$

this leaves us with x in the denominator, but only x^2 in the numerator – we can't also keep the x. Since $x^2/x = x$, we can conclude that for large $|x|$ we have a linear asymptote with gradient 1; i.e., $y = 1x + c$. However, the large $|x|$ completely dominates the constant c, and so this informal reasoning about the asymptote cannot correctly identify the specific value of c. Similarly, in

$$\frac{x + 1 - \frac{1}{x}}{1 - \frac{1}{x}},$$

when we take $|x|$ to be large we must discard everything but the x in the numerator.

Just teaching students the correct way to find an oblique asymptote might not be sufficient, as there is always the risk that they 'discover' this wrong method. So, I think it is important to make them aware of it, and why it doesn't work (French, & Stripp, 1997, pp. 7-8).

If the numerator is more than 1 degree higher than the denominator, our asymptote won't just be at an angle to the axes; it won't even be a straight line! Consider a function like

$$y = \frac{x^3 + 3x + 1}{x}.$$

Dividing out the x, we get $y = x^2 + 3 + \frac{1}{x}$. Clearly, as $|x|$ goes to infinity, y will get arbitrarily close to the parabola $y = x^2 + 3$ (Figure 9).

I think that by playing around with different possibilities of asymptotes the topic can become less procedural and the connections can become more salient.

Acknowledgement

I would like to thank Chris Sangwin for very helpful discussions about this article.

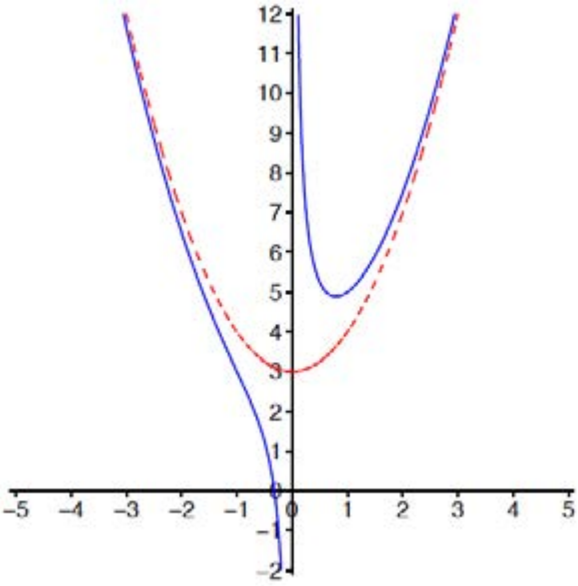


Figure 9. The graph of $y = \frac{x^3+3x+1}{x}$ (blue) when zoomed out looks like the graph of $y = x^2 + 3$ (red).

References

Foster, C. (2020). 'Trusting in patterns', *Mathematics in School,* 49(3), pp. 17–19.

French, D., & Stripp, C. (1997). *'Pig' and other tales: A book of mathematical readings with questions and specimen answers for students of A level or Scottish Highers*, Mathematical Association.

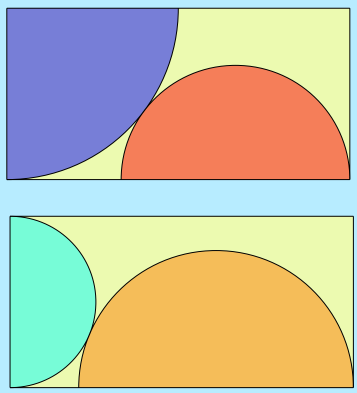
Author: Prof. Colin Foster, Dept. of Mathematics Education, Schofield Building, Loughborough University, Loughborough LE11 3TU.
Email: c@foster77.co.uk
Website: www.foster77.co.uk

Two Domino Problems

The dominoes to the right are both 2 by 4.

Drawn inside the upper domino is a quadrant and a semicircle,
while the lower domino contains two semicircles.

Find the radii of the four inscribed shapes.



Solutions are given on pages 69 and 76.

Chris Pritchard