Unconvincing Proofs: The Harmonic Series

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Some famous, perfectly valid proofs may nevertheless seem quite unconvincing at their first viewing. I am not just talking about the *result* being surprising, but the *method* of proof having a kind of dubious air about it. As our example, let's take the divergence of the harmonic series,

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\cdots$$

Theorem: The harmonic series diverges.

There is no doubt that this is a surprising *result*. As Havil commented (2003, p. 22), "No property is more unexpected than H_n's divergence". We have to battle against the familiarity of the result and its proofs (our 'curse of knowledge') if we're to see how shocking this might (should) be to a student meeting it for the first time. It seems natural to assume that a series must converge if the terms get 'smaller and smaller'. Experience with geometric series, such as

$$\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k=1,$$

and the associated pictures, such as a unit square, in which we successively colour half and then half of what's left, and so on, lead us to expect that so long as the terms *decrease* in size, the sum is bound to get 'closer and closer' to some value which it cannot ever exceed.

But 'closer and closer' is not what convergence is about (Foster, 2018): to converge, the sum needs to get *arbitrarily close* to a fixed number (i.e., closer than any real number you can think of, however small), and stay there – and the harmonic series never does that. The harmonic series diverges to infinity, meaning that you can choose any number, as large as you like, and eventually the sum will exceed that number – although you might need to take a very large number of terms to get past it.

So, it's undoubtedly a surprising *result*. But what I'm thinking about here is the perhaps unconvincing nature of some *proofs* of this, when seen from a student's perspective. A great many proofs that the harmonic series diverges have been given (see Kifowit & Stamps, 2006; Kifowit, 2019), but the most common one is undoubtedly that due to Nicole Oresme (1323–1382). And I think that thoughtful students don't always find this proof particularly convincing.

Oresme's proof involves putting the terms of the harmonic series into larger and larger groups, and noting that each group has a sum that is greater than $\frac{1}{2}$:

$$\sum_{k=1}^{n}\frac{1}{k}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\cdots$$
$$=1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\cdots$$

Looking at the first bracket, since $\frac{1}{3}>\frac{1}{4}$, it follows that $\frac{1}{3}+\frac{1}{4}>\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$. Similarly, in the second bracket, $\frac{1}{5}$, $\frac{1}{6}$ and $\frac{1}{7}$ are each separately greater than $\frac{1}{8}$, so it follows that

$$\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}>\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{1}{2}.$$

We can continue grouping terms like this, so that the nth bracket will contain 2^n fractions, each of which is greater than or equal to $\frac{1}{2^{n+1}}$, so the total for the nth bracket will necessarily be greater than $\frac{1}{2}$. This means that

$$\sum_{k=1}^{n}\frac{1}{k}>1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots,$$

and we know that the right-hand side is unbounded, so the sum of the harmonic series is greater than any real number.

There is nothing wrong with this proof, and I am not suggesting that there is. However, to many students it can feel rather dodgy, because of the fact that we are including *more and more terms* in each group. Granted, there are infinitely many terms available, so we will certainly never run out of terms. But, it may still feel a little dubious to be having to take bigger and bigger bites out of the series in order to find our $\frac{1}{2}$s. Is this really legitimate? Couldn't you play this trick on series that actually *do* converge, and use this method to prove that they *don't*? Unless you try (and, we hope, fail!) to do this, I think it's hard to claim that you really understand Oresme's proof. So let's do this now.

Reassuringly, Oresme's approach definitely *doesn't* work for the convergent geometric series:

$$\sum_{k=1}^{n}\left(\frac{1}{3}\right)^k=\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\frac{1}{3^4}+\cdots$$

This is an easy case, because *every* term in this series is larger than the sum of all the subsequent terms. For example,

$$\frac{1}{3} > \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \cdots$$

So, even if you include *all* the terms after the first term, $\frac{1}{3}$, you can never make another bracket that is as big as $\frac{1}{3}$, so you can never get to

$$\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \cdots,$$

because you can never even get to $\frac{2}{3}$. This corresponds to the fact that the infinite sum is $\frac{1}{2}$, which is less than $\frac{2}{3}$. And, starting with a later term doesn't help, as exactly the same thing will happen. So, we have failed to use Oresme's method to prove the divergence of $\sum_{k=1}^{n} \left(\frac{1}{3}\right)^k$, which is good!

But, what if we picked some *smaller* fraction than $\frac{1}{3}$, say something really small like $\frac{1}{3^{10}}$, aiming to eventually get to the divergent series

$$\frac{1}{3^{10}} + \frac{1}{3^{10}} + \frac{1}{3^{10}} + \cdots \ ?$$

Is it obvious that this wouldn't work? It's one of those things that becomes obvious if you think about it for long enough! The process seems to start off successfully (for our attempt to prove divergence), because $\frac{1}{3^2} > \frac{1}{3^{10}}$ and $\frac{1}{3^3} > \frac{1}{3^{10}}$, and so on. But, eventually (in this case, for all $n > 10$), we can see that $\frac{1}{3^n} < \frac{1}{3^{10}}$. So, to continue to make brackets containing terms that sum to more than $\frac{1}{3^{10}}$, we would need to add up

$$\frac{1}{3^{11}} + \frac{1}{3^{12}} + \frac{1}{3^{13}} + \cdots$$

until this sum exceeds $\frac{1}{3^{10}}$. And we can see that that will never happen, since we have already noted that every term in this series is larger than the sum of all the subsequent terms. So, this approach is doomed to fail, and choosing a larger n in $\frac{1}{3^n}$ won't help. This means that, however small a fixed number we choose for the total of our brackets, eventually, far enough out, the terms of the series will drop below this value, and then the sum of all the *remaining* terms can never match it, and so we are doomed. Doomed is good, of course, since we don't really want to prove that convergent series like $\sum_{k=1}^{n} \left(\frac{1}{3}\right)^k$ diverge!

However, it is certainly *not* the case for all convergent series, even for all convergent *geometric* series, that each term has to be larger than the sum of all the subsequent terms. In this respect, the series $\sum_{k=1}^{n} \left(\frac{1}{3}\right)^k$ was an easy case. In general,

$$\sum_{k=0}^{n} ar^k = \frac{a}{1-r} \text{ for } |r| < 1$$

and $$\sum_{k=1}^{n} ar^k = \frac{ar}{1-r}.$$

So, each term will be larger than the sum of all the subsequent terms iff

$$a > \frac{ar}{1-r}.$$

Now, since $1 - r > 0$, and assuming that $a > 0$, we can rearrange to give $1 - r > r$; so $0 < r < \frac{1}{2}$. So, it worked out this way for $\sum_{k=1}^{n} \left(\frac{1}{3}\right)^k$ because r was equal to $\frac{1}{3}$, which is less than $\frac{1}{2}$. The boundary case, when $r = \frac{1}{2}$, is the equality

$$\sum_{k=2}^{n} \left(\frac{1}{2}\right)^k = \frac{1}{2},$$

and, for all of the geometric series for which

$$\tfrac{1}{2} < r < 1,$$

even though they converge, the total of all the subsequent terms is *more than* the term that came before them, so it's less obvious that Oresme's approach must fail with them.

Let's take, for example,

$$\sum_{k=1}^{n} \left(\frac{2}{3}\right)^k = 2,$$

and we note that

$$\sum_{k=2}^{n} \left(\frac{2}{3}\right)^k = \frac{4}{3} > \frac{2}{3}.$$

So, we can see, even just from among geometric series, that each term being larger than the sum of all the remaining ones cannot be a *necessary* condition for convergence. So, let's try Oresme's method on something like $\sum_{k=1}^{n} \left(\frac{2}{3}\right)^k$. Can we prove that this convergent series diverges? Hopefully not!

We write:

$$\sum_{k=1}^{n} \left(\frac{2}{3}\right)^k = \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^4 + \left(\frac{2}{3}\right)^5 + \cdots$$

The task is to see if we can collect these terms into larger and larger groups such that each group exceeds $\frac{2}{3}$. To begin with, we have no difficulty. By inspection, we can see that

$$\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 = \frac{20}{27} > \frac{2}{3}.$$

So, we move on to $\left(\frac{2}{3}\right)^4$. How many subsequent terms do we need to add to this to exceed $\frac{2}{3}$? Now, we have a problem, because even if we take *all* of the remaining terms, the sum will be

$$\frac{\left(\frac{2}{3}\right)^4}{1-\frac{2}{3}} = \frac{16}{27} < \frac{2}{3}.$$

This doesn't *prove* that the series converges, but it does mean that Oresme's method cannot prove that it doesn't.

Would it work if we picked some smaller comparison number than $\frac{2}{3}$? Let's try $\frac{1}{9}$. We want to group the terms of

$$\sum_{k=1}^{n}\left(\frac{2}{3}\right)^k = \frac{2}{3}+\left(\frac{2}{3}\right)^2+\left(\frac{2}{3}\right)^3+\left(\frac{2}{3}\right)^4+\left(\frac{2}{3}\right)^5+\cdots$$

into brackets such that each is greater than $\frac{1}{9}$. Can we do it? We see that $\left(\frac{2}{3}\right)^n > \frac{1}{9}$, for $n < 6$, so we don't need any brackets yet for those terms, as each one individually exceeds $\frac{1}{9}$. Then, $\left(\frac{2}{3}\right)^6 + \left(\frac{2}{3}\right)^7 > \frac{1}{9}$, so we have a group of two terms. And then we find that the terms 8 to 14 have a sum greater than $\frac{1}{9}$, so now we have a group of 7 terms. But *then* we are stuck, because the infinite sum from term 15 onwards is $< \frac{1}{9}$.

Why exactly does Oresme's method seem to always (eventually) fail for convergent series? This may be obvious depending on how you understand convergent series. For a series to converge, no matter how small a number we think of, we can always get closer to a fixed value than that small number, provided we just take enough terms. The idea of *Cauchy convergence* is that, provided you go far enough out, the magnitude of *any sum of consecutive terms* will be less than any arbitrary number we can think of, however small.

It seems to me that Oresme's proof raises important issues to think about, but is nowhere close to the easiest way to see that the harmonic series diverges. I think I much prefer an alternative approach (this is proof #6 in Kifowit & Stamps, 2006, where they credit Honsberger and also Gillman, but it has also been rediscovered by Goldmakher (n.d.) – see also Havil, 2003, pp. 22–23). In this much simpler approach, we first group the terms in *pairs* throughout – there is no taking bigger and bigger bites. Then, to create an inequality, we replace the *larger* term in each pair with a copy of the *smaller* term in that pair, and observe that this recreates the original series:

$$\begin{aligned} H &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\cdots \\ &= \left(1+\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}\right)+\left(\frac{1}{7}+\frac{1}{8}\right)+\cdots \\ &> \left(\frac{1}{2}+\frac{1}{2}\right)+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{6}+\frac{1}{6}\right)+\left(\frac{1}{8}+\frac{1}{8}\right)+\cdots \\ &= H \end{aligned}$$

The contradiction $H > H$ tells us that the series cannot sum to any finite value H, and hence that it diverges.

Although this proof relies on the supposedly difficult method of 'proof by contradiction' (but see Foster, 2021), I think it raises far fewer problems for thoughtful students. But I would be very interested to know what other readers think.

References

Foster, C., 'Questions pupils ask: Is calculus exact?', *Mathematics in School,* 47, 3 (May 2018), 36–38.

Foster, C., 2021 'Making sense of proof by contradiction', *Scottish Mathematical Council Journal* 51 (2021), 74–77.

Havil, J., *Gamma: Exploring Euler's constant*. Princeton University Press, 2003.

Goldmakher, L. (n.d.). *A Short(er) proof of the divergence of the harmonic series*. https://web.williams.edu/Mathematics/lg5/harmonic.pdf

Kifowit, S. J., *More proofs of divergence of the harmonic series* (2019), https://stevekifowit.com/pubs/harm2.pdf

Kifowit, S. J., & Stamps, T. A., 'The harmonic series diverges again and again', *The American Mathematical Association of Two-Year Colleges Review* 27, 2 (2006), 31–43, http://scipp.ucsc.edu/~haber/archives/physics116A10/harmapa.pdf

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