Unconvincing proofs: The sum of a geometric sequence

Colin Foster

The proof of the formula

$$S = \frac{a}{1-r}$$

for the sum S of a geometric sequence with first term $a \neq 0$ and common ratio $|r| < 1$ is usually done by multiplying the sum by r and subtracting:

$$\begin{aligned} S &= a + ar + ar^2 + ar^3 + \cdots \\ rS &= ar + ar^2 + ar^3 + ar^4 + \cdots \\ S - rS &= a \\ S\,(1-r) &= a \\ S &= \frac{a}{1-r}. \end{aligned}$$

But this proof can feel unconvincing to students in the subtraction step. We can match up the terms that cancel out, but it looks a bit as though the rS sum has *one more term* than the S sum:

$$\begin{aligned} S &= a + ar + ar^2 + ar^3 + \cdots \\ rS &= ar + ar^2 + ar^3 + ar^4 + \cdots \end{aligned}$$

What do we pair off ar^4 with?

Of course, we can include an ar^4 term in the S sum if we want to, but then students will protest that we are stacking the deck, because we have *five* terms of S but only *four* terms of rS. If we want to include ar^4 in S, then they may think that we should surely include ar^5 in rS, and then we are back to the original mismatching problem.

The teacher might be inclined to wave this issue away by saying that these series are *infinite*, so there will always be a term in S to match *any* term we care to think of in rS. If we doubt it, then we should try to say which term it is that it will first go wrong for, and we will not be able to. However, I think it may still feel to thoughtful students as though something fishy is going on within the mysterious '...' parts of these sums, and that there is something happening here that they don't fully understand. If they nevertheless shrug and allow the teacher to move on, I think something is lost in terms of their sense that mathematics is rigorous and meaningful. But what to do?

Finite sums

One response is to say that we are being a bit casual with our infinite series here, and we would be on much safer ground if we dealt with *finite* series instead, where the murkiness of the '...' regions doesn't arise.

So, instead we write

$$\begin{aligned} S_n &= a + ar + ar^2 + ar^3 + \cdots + ar^{n-2} + ar^{n-1} \\ rS_n &= ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1} + ar^n \\ S_n - rS_n &= a - ar^n \\ S_n\,(1-r) &= a(1-r^n) \\ S_n &= \frac{a\,(1-r^n)}{1-r} \qquad \text{(Note1).} \end{aligned}$$

Now, with a greater or lesser degree of formality, we can say that, iff $|r| < 1$, then as n gets large we can 'forget about' the r^n, as it will be 'negligible' or 'vanishingly small', or some such language. We could make this look more formal by using $\rightarrow$ notation and the language of 'null sequence', and writing:

As $n \rightarrow \infty$, $r^n \rightarrow 0$,

$$\frac{a\,(1-r^n)}{1-r} \rightarrow \frac{a}{1-r},$$

and then we write S_∞ as just S.

We have to try to help students see that this *isn't an approximation*, or, if it is, then it is a *perfect* approximation, because the 'error' can be made as small as we like, just by making n large enough. There is no positive real number that the error can't be made smaller than, simply by choosing a large enough n. That may still not sound 'exact' to the students' ears, though. They are likely to say that, for complete accuracy, they want r^n to be *precisely equal to zero*, not just to 'tend' in that general sort of direction (see Foster, 2018).

Proofs Without Words

An alternative method of proof is to use one of the *Proofs without words* (often, in practice, 'with only few words'). Consider the example shown in the diagram (due to J. H. Webb, taken from Nelson, 2003, p. 119):

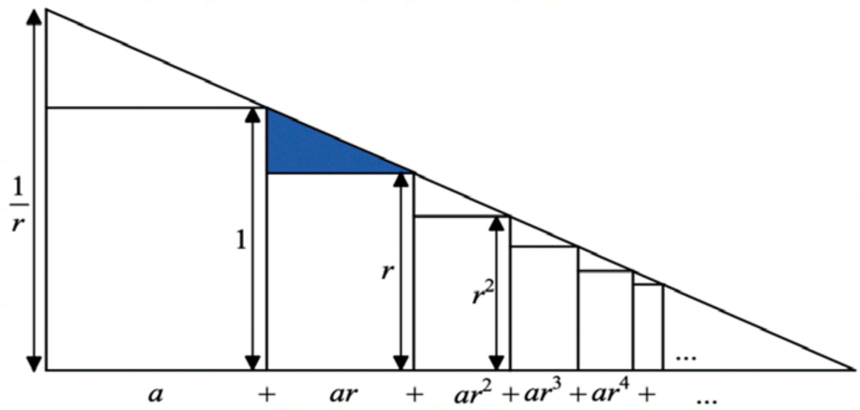
Moving from left to right, each successive right-angled triangle has both its base and its height multiplied by r. So, all of the right-angled triangles are similar. Equating the ratios of the base to height of the largest one and the blue one, we obtain:

$$\frac{S}{\left(\frac{1}{r}\right)} = \frac{ar}{1-r},$$

so,

$$S = \frac{a}{1-r}.$$

Proofs *without words* can be quick and convincing, but can often feel like a rabbit pulled out of a hat. Where they relate to limits, they often suffer from the problem that the really interesting bit, that you are potentially worried about, tends to be happening in the corner part of the diagram that is so small that you can't see clearly what is going on!



A proportionality approach

What follows is certainly no more rigorous than any of the above, but is perhaps easier for students to swallow.

Let S, as before, be the geometric series with first term $a \neq 0$ and common ratio $|r| < 1$:

$$S = a + ar + ar^2 + ar^3 + \cdots.$$

It is clear that S must be *proportional* to a, since every term in the series is proportional to a. (If a suddenly becomes 10 times as big, then S will also become 10 times as big.)

So, we can write

$$S = ka,$$

where k is a constant of proportionality *depending only on* r.

Removing the first term, a, from this series leaves *another* geometric series, with the same common ratio r, and with first term ar, instead of a:

$$S - a = ar + ar^2 + ar^3 + \cdots$$

Note that here we are just subtracting *one term*, a, not, as before, an entire series.

Now, looking at this new geometric series on the right-hand side, by the same argument as before, this series must be proportional to ar, with the same constant of proportionality, k:

$$S - a = kar.$$

So,

$$ka - a = kar.$$

Since $a \neq 0$,

$$k - 1 = kr$$
$$k(1-r) = 1$$
$$k = \frac{1}{1-r}.$$

So,

$$S = \frac{a}{1-r},$$

the standard formula.

I appreciate that this approach suffers from its own sleight of hand, where we subtly assume that infinite sums can be scalar multiplied. But overall I think avoiding the telescoping of the infinite series can be helpful. Subtracting a single constant term, a, from an infinite series is much less of a potential difficulty than trying to find the difference between two infinite sums.

Note

1. Here, the '...' sections have a quite different meaning. We are no longer saying 'and so on like this forever'. Now, we are merely omitting a *finite* number of terms that we just can't be bothered to write out. For any particular n, we could in principle write them all out, so the ellipsis here is just a convenience, rather than a necessity. We can say exactly what each of the terms hidden in the ellipsis actually is.

References

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Nelson, R. B. (2003). *Proofs without words: Exercises in visual thinking*. Mathematical Association of America.

Author

Colin Foster, Department of Mathematics Education, Schofield Building, Loughborough University, Loughborough LE11 3TU.
Email: c@foster77.co.uk
website: www.foster77.co.uk
blog: https://blog.foster77.co.uk/

Some Sums

Adam McBride

Introduction

We shall consider several problems related to expressing positive integers as sums of positive integers of certain types, including squares and triangular numbers. The concepts involved are elementary and most of the problems could form the basis of investigations in the classroom.

1s and 2s

We start with the following little puzzle.

In how many ways can the number 12 be expressed as a sum of 1s and 2s ?

For example, we can write 12 as

$2 + 2 + 2 + 2 + 2 + 2$

$2 + 1 + 2 + 1 + 2 + 1 + 2 + 1$

$2 + 2 + 2 + 2 + 1 + 1 + 1 + 1.$

Although the last two sums contain the same ingredients, namely four 1s and four 2s, they are to be regarded as different. In other words, order matters.

A natural response to the question might be "Quite a few" but, although 12 is not a large number *per se*, it is large enough in the present context to make the answer far from obvious immediately.

In such a situation it is a good idea to look at smaller numbers first to try to detect what is going on. Starting from 1, we obtain the following sums.

$1 = 1$

$2 = 1 + 1;\; 2$

$3 = 1 + 1 + 1;\; 2 + 1;\; 1 + 2$

$4 = 1 + 1 + 1 + 1;\; 2 + 1 + 1;\; 1 + 2 + 1;\; 1 + 1 + 2;\; 2 + 2.$

In total we have 1, 2, 3 and 5 sums respectively. At this stage, students might make the conjecture that we are dealing with Fibonacci numbers and checking what happens for 5 and 6 seems to confirm this. They could then be challenged to try to *prove* this conjecture.

The proof is relatively simple and elegant. To get a sum of 1s and 2s for a positive integer n, we note that such a sum ends with either a 1 or a 2 and therefore we

either

add a 1 to one of the sums for the integer $n-1$

or

add a 2 to one of the sums for the integer $n-2$.

If we denote by $f(n)$ the number of ways of writing n as a sum of 1s and 2s we therefore have

$$f(n) = f(n-1) + f(n-2);\; f(1) = 1;\; f(2) = 2.$$

We recognise the recurrence relation for the familiar Fibonacci sequence (albeit with a single 1 at the start rather than the usual two 1s). It is now straightforward to work out that the answer to our puzzle is $f(12) = \mathbf{233}$, quite a few indeed.

Fibonacci Numbers have been around for over 800 years and it might be thought that everything that could be said about them has been said. Yet they keep turning up in all sorts of contexts. For example, try working out the number of ways of paving a