COUNTING ZEROES

Here's a trick question:

There are 6 zeroes in a million.

How many zeroes are there in half a million?

The idea is that you are supposed to immediately give the response "3" without thinking too hard. Of course, when you think about it more carefully, you realise that half a million is 500 000, which has five zeroes. So, the number of zeroes has decreased by one rather than halved. Why does this happen?

You can relate this to logarithms (to base 10), which decrease by 1 when you divide the number by 10. So, for instance, 10^6 has 6 zeroes, and $\frac{10^6}{10}=10^5$, which has 5 zeroes. However, in our original question we were dividing by 2, not by 10, so why does this still work? Will it always work?

Let's explore what happens to the number of zeroes if you keep on halving:

	Number	Number of zeroes
	1 000 000	6
↓ ÷2	500 000	5
↓ ÷2	250 000	4
↓ ÷2	125 000	3
↓ ÷2	62 500	2
↓ ÷2	31 250	1
↓ ÷2	15 625	0

We can see that every time we halve, we lose another zero, until we have no zeroes left. Does this always work, starting with any number?

Clearly not. For example, 60 has 1 zero, and when you halve it you get 30, which also has 1 zero. So, the number of zeroes has stayed the same. Can you halve a number and *increase* the number of zeroes?

The answer is yes, provided that the zeroes don't have to all be at the end of the number! For example, $\frac{2120}{2}=1060$, so the number of zeroes here goes from 1 to 2. Can you find a *smaller* number where this happens? Can you find a number where the number of zeroes increases by *more than 1* when you halve it?

If a number ends in n zeroes, then it must be a multiple[1] of 10^n, which means that it must be a multiple of 2^a5^b, where a and b are positive integers. Here, $\min(a, b) = n$, the number of zeroes on the end. This is because we get zeroes from powers of 10, and each 10, when prime factorised, must be made up of the product of a 2 and a 5. So, the power of 10 that we get will be limited by whichever power of 2 or power of 5 is smaller.

Now if we divide our number by 2, what will happen? If $0 < a \le b$, then $\min(a, b)$ will become $n - 1$, and we will lose a zero from the end. This is what happened above with $10^n = 2^n5^n$, where $a = b$. On the other hand, if $a > b > 0$, then when we divide the number by 2 we can see that $\min(a, b) = n$ will *not* change, so the number of zeroes on the end will stay the same. This is why, for example, $\frac{2000}{2}=1000$, and there are still three zeroes on the end. In 2000 we had more factors of 2 than factors of 5, so we could afford to lose a factor of 2 without reducing the number of factors of 10.

This analysis, by splitting the factors of 10 into 2s and 5s, also reveals that the same thing will happen why you divide a number by 5:

	Number	Number of zeroes
	1 000 000	6
↓ ÷5	200 000	5
↓ ÷5	40 000	4
↓ ÷5	8000	3
↓ ÷5	1600	2
↓ ÷5	320	1
↓ ÷5	64	0

What other questions can you ask about zeroes in numbers?

Note

1 Here I mean a multiple that does not include any powers of 2 or 5.

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