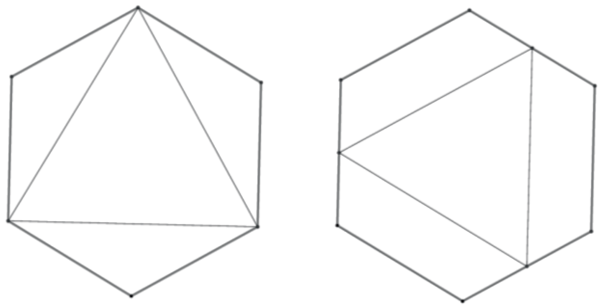
EQUILATERAL TRIANGLES WITHIN A REGULAR HEXAGON

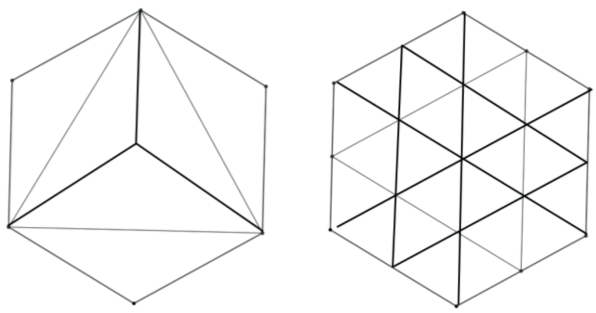
In each of the two diagrams below, equilateral triangles fit exactly inside identical regular hexagons. In the first diagram, the triangle's vertices coincide with three of the vertices of the hexagon. In the second diagram they lie at the midpoints of three of the sides of the hexagon.

Which triangle do you think is bigger? Why?

How much bigger is it? Why?

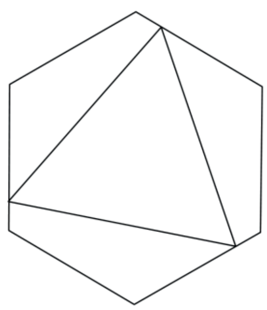


You may be able to see just by looking that the first triangle is bigger than the second one. You could use trigonometry to find the areas, or use 'dissection' methods, such as those shown below.



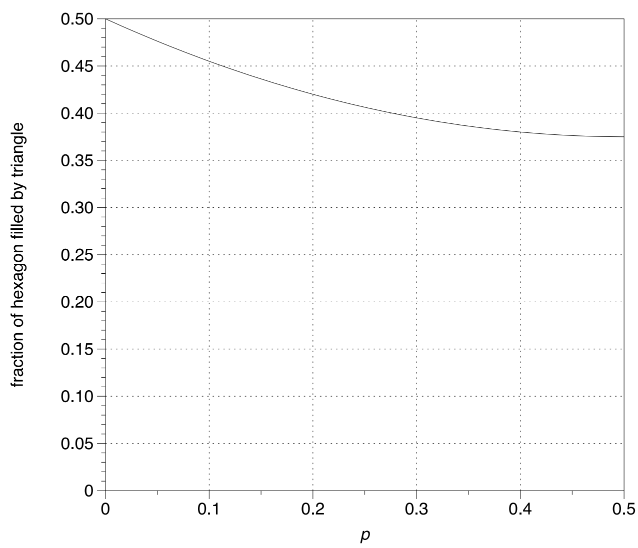
In the first diagram, by 'folding' in/out isosceles triangles, you can see that the triangle is $\frac{1}{2}$ of the area of the hexagon. In the second case, by counting the little equilateral triangles, the big equilateral triangle turns out to be $\frac{9}{24} = \frac{3}{8}$ of the area of the surrounding hexagon.

Now, what about in-between cases like this?



This can be analysed using the cosine rule and the $\frac{1}{2}ab\sin C$ area formula. If the triangle has its vertices a fraction p of the way along the sides of the hexagon (where $0 \le p \le \frac{1}{2}$), then the area of the triangle turns out to be equal to $\frac{1}{2}(p^2 - p + 1)$ of the area of the hexagon. We can check that, when $p = 0$, this gives an answer of $\frac{1}{2}$ (the first case) and, when $p = \frac{1}{2}$, we obtain $\frac{3}{8}$ (the second case).

You might like to set up quadratic equations, such as $\frac{1}{2}(p^2 - p + 1) = \frac{5}{2}$, to find positions that lead to a specified area ratio, and you could use GeoGebra for checking. The graph of $y = \frac{1}{2}(p^2 - p + 1)$ below shows the behaviour across the range of possible values of p.



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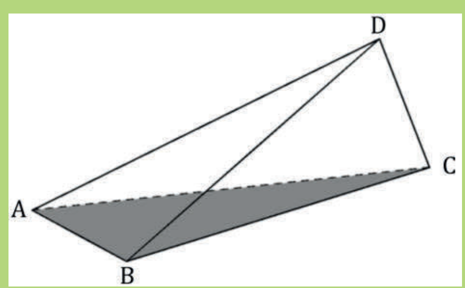
NOT-SO-QUICKIE 3



A very small drinks carton is formed by arranging four congruent triangles as shown. $AC = AD = BC = BD = 9$ cm and $AB = CD = 12$ cm. Show that the volume is 72 ml.