SUMS OF PAIRS

I am thinking of four numbers. When I add every possible pair of numbers, I get the six answers: 2, 3, 4, 5, 6 and 7. What might my four numbers be?

You could approach this problem by trial and improvement – otherwise known as “by inspection”. (For some reason when people want to imply that it is easy they tend to call it “by inspection”!) If we assume that my four numbers are non-negative integers, then we can think about the possible partitions of the sums (without regard for order):

$2=0+2$ or $1+1$

$3=0+3$ or $1+2$

$4=0+4$ or $1+3$ or $2+2$

$5=0+5$ or $1+4$ or $2+3$

$6=0+6$ or $1+5$ or $2+4$ or $3+3$

$7=0+7$ or $1+6$ or $2+5$ or $3+4$

The smaller numbers have fewer partitions, so are easier to begin with. If the six numbers I gave you are all different, then that implies that my four numbers must also be all different. (Is the converse true? If my four numbers are all different does that mean that the six sums will be also?) You will discover that the 1 is not actually as useful as it might first appear to be!

When you have your solution, you need to ask yourself whether it is unique. Might there be another set of four numbers which would give these same six sums? It might help to look at the six numbers and to think about what things have to be true for any set of six numbers generated in this way. For example, the set of numbers 1, 2, 5, 6, 7 and 8 could *not* be obtained as the sum of every pair of a set of four numbers. Can you see why not?

Another way to get started would be to choose your own set of four numbers and generate the six sums of pairs and look for connections between the sums you obtain and the original numbers. If you find this too hard you could reduce your starting set of numbers to three. This might remind you of *arithmagons* (Foster, 2014).

If we represent the four starting numbers as a, b, c and d then our six sums will be

$s_{ab}=a+b$

$s_{ac}=a+c$

$s_{ad}=a+d$

$s_{cd}=c+d$

$s_{bd}=b+d$

$s_{bc}=b+c$

The first thing to notice is that there are six equations but only four unknowns. So we might know enough to determine a, b, c and d uniquely. However, some combinations of these equations are clearly equivalent to others, so we don't have quite as much information as we might think. If we look carefully, we can see that

$$s_{ab} + s_{cd} = s_{ad} + s_{bc}.$$

So a necessary condition for six numbers to be generated as the sums of pairs of four numbers is that the six numbers themselves come in three pairs of two, each with the same sum. Checking back with our original set of six numbers, we notice that $2+7=3+6=4+5$, and the set of numbers 1, 2, 5, 6, 7 and 8 that I claimed was impossible cannot be split up in this way, because the total is 29, and since these numbers are integers no pair of them will sum to $\frac{29}{3}$.

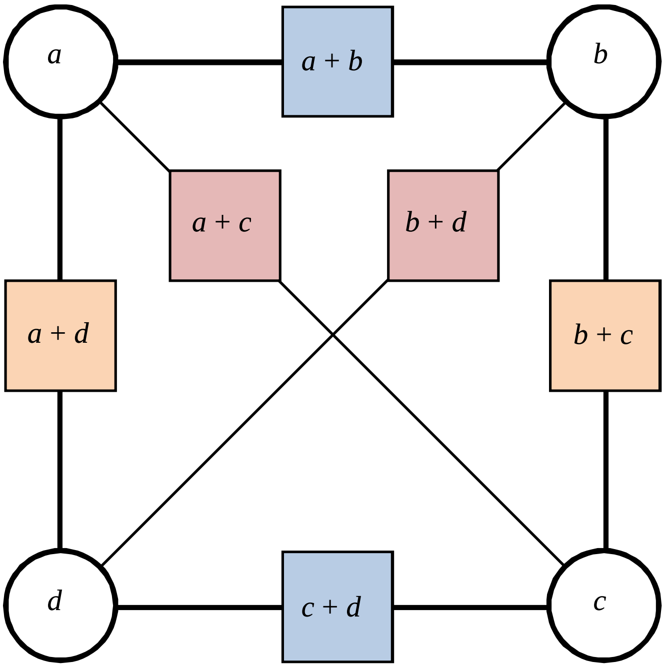
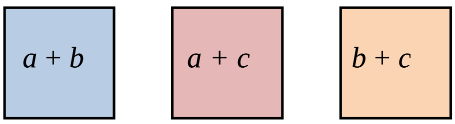
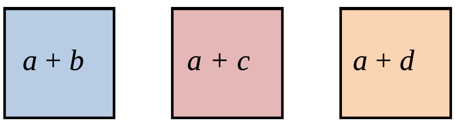


Figure 1.

One way to visualise what we are doing is to represent our four numbers and six sums in an arithmagon-type diagram (Figure 1). We can arrange our six sums in three pairs, each having the same total, as shown by the three colours in Figure 1. Then we can solve the problem by choosing one blue, one pink and one green sum to make an arithmagon triangle. However, we need to be careful – just any choice of a blue, a pink and a green may not form an arithmagon triangle. Without loss of generality we can pick any blue and any pink, but then there are *two* choices for green, only one of which forms the third side of an arithmagon. For example,



makes an arithmagon triangle, but



does not, because the three lines containing these boxes all meet at circle a, and so do not form a triangle.

This gives us a way to solve the problem. If we return to our original six numbers, and again split them into pairs with equal sums, $2+7=3+6=4+5$, then we can choose either number from each of the first two pairs, but then we must try *both* numbers from the third pair (the green 4 or 5 in Figure 2). The purple arithmagon triangle can then be solved (twice) in the standard way (Foster, 2014). Trying the green 5 first, we know that the sum of the three box numbers $2 + 3 + 5 = 10$ must be equal to twice the sum of the three circle numbers, since the sum of the three boxes includes each circle number twice, so the three circle numbers must sum to 5. Since the top line indicates that a and b sum to 2, this means that d must be 3, and so we obtain the solution: $a = 2$, $b = 0$, $d = 3$. We now know everything, since if our green number is 5 the other green number must be 4, so $b + c = 4$, meaning that $c = 4$, and we end up with the answer that the original four numbers are 0, 2, 3 and 4. Were those the numbers you obtained?

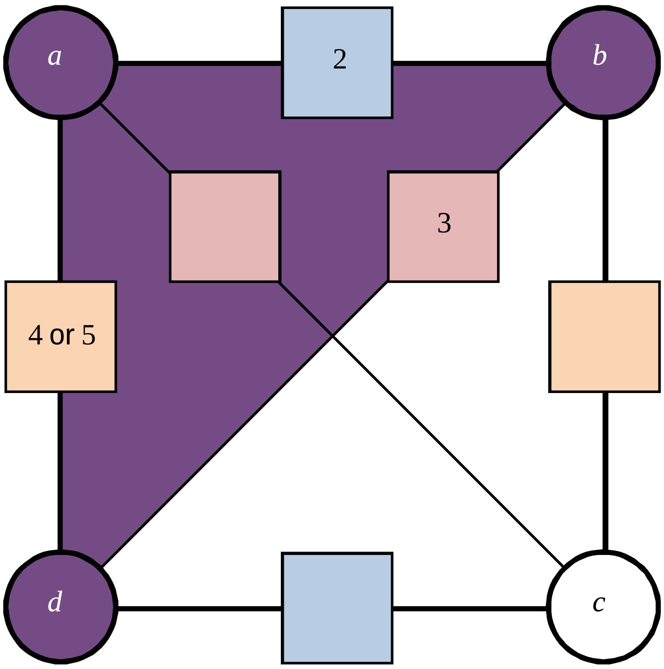


Figure 2.

We now go back and try having the green 4, rather than the 5, between a and d in Figure 2. Now $2 + 3 + 4 = 9$, so the three circles must add up to 4.5. Since $a + b = 2$, this means that $d = 2.5$, and so $a = 1.5$, $b = 0.5$ and we find that $c = 4.5$. So the set of four numbers 0.5, 1.5, 2.5 and 4.5 is also a solution – and harder to find by inspection than 0, 2, 3 and 4.

There are lots of questions you might ask at this point. Here are some of mine:

- What four starting numbers produce the six sums 1, 2, 3, 4, 5, 6?
- Will there always be two sets of possible starting numbers?
- What happens if you begin with more than four numbers?
- What happens if I give you the *differences* of pairs rather than the sums of pairs?

Reference

Foster, C. (2014). Arithmagons. *Teach Secondary*, *3*(2), 57–59. Freely available at: http://www.teachsecondary.com/zipfiles/lesson-plan-ks3-maths-arithmagons.pdf

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