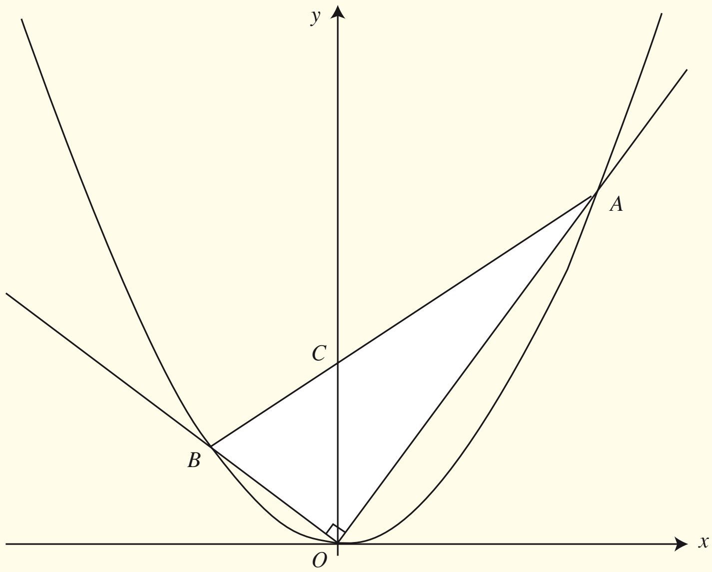
TRIANGLES AND PARABOLAS ~ the hard part!

We left you with a problem last time – for those who know about finding areas under curves, or between curves and straight lines. Remember that *OA* has slope *m* and *OB* is at right angles to it. For what values, if any, of *m* does the triangle *AOB* take up half the area which *AB* cuts off from the parabola $y = x^2$?



This is equivalent to saying that the area between *OB* and the curve plus the area between *OA* and the curve is equal to the area of the triangle *AOB* which we calculated last time as being $\frac{m^2+1}{2m}$. $A = (m, m^2)$; $B = \left(-\frac{1}{m}, \frac{1}{m^2}\right)$.

Hence we have the following:

$$\int_{-\frac{1}{m}}^{0}\left(-\frac{x}{m} - x^2\right)\mathrm{d}x + \int_0^m (mx - x^2)\,\mathrm{d}x = \frac{m^2+1}{2m}$$

$$\therefore \quad \left[-\frac{x^2}{2m} - \frac{x^3}{3}\right]_{-\frac{1}{m}}^{0} + \left[\frac{mx^2}{2} - \frac{x^3}{3}\right]_0^m = \frac{m^2+1}{2m}$$

$$\therefore \quad \frac{1}{2m^3} - \frac{1}{3m^3} + \frac{m^3}{2} - \frac{m^3}{3} = \frac{m^2+1}{2m}$$

Multiply by $6m^3$ and collect up terms:

$$1 + m^6 = 3m^2(m^2+1)$$

We now make the substitution $p = m^2$ to create the cubic equation $p^3 - 3p^2 - 3p + 1 = 0$.

This factorises to give $(p+1)(p^2 - 4p + 1) = 0$, giving $p = -1$ (impossible, as $p = m^2$) or $p = 2 \pm \sqrt{3}$.

Hence $m = \pm\sqrt{2+\sqrt{3}}$ or $m = \pm\sqrt{2-\sqrt{3}}$. But we claimed last time that there was essentially one solution geometrically, so how can we prove that?

If we multiply the positive value of one possibility, by the negative value of the other we get $-\sqrt{(2+\sqrt{3})(2-\sqrt{3})}$, but this simplifies to -1, so that if one value of *m* corresponds to the line *OA*, the appropriate other value gives *OB*, and vice versa. The other values give the mirror images in the *y*-axis, so essentially there is, geometrically, a unique solution.

Colin Foster
[Colin teaches at King Henry VIII School, Coventry]

OUT OF THE ARK

Well, not quite, but from a long time ago. Jan has been clearing out again, and came across some School Certificate papers from 1941 – probably before most of your parents were born! This was before GCSE, or even O levels. There were three papers, each of 2½ hours, one on Arithmetic, one on Algebra and one on Geometry. No calculators, of course, but logarithm tables were allowed for all but the first part of the Arithmetic paper. Do you remember that we looked at finding square roots 'by hand' not long ago? Then you could do question 2(ii): Find the square root of 436.281 correct to three significant figures. I'll save you from the question about mixing different qualities of margarine and butter (all in imperial units and £ s d of course), but how about this one on percentages?

A man bought 144 exactly similar articles and marked each article for sale at a profit of 40% on its original cost. After selling 100 articles at this price, he altered the selling price of the remaining articles so that each should gain 5% of its original cost, and he sold 40 articles at this price. Find what percentage loss on its original cost he sold each of the four remaining articles, if on his whole outlay he made a profit of $28\frac{1}{3}$ %.

Trisha Haslam