[MATHS PROBLEM]

ADDING SURDS

Colin Foster looks at the different ways in which surds can be combined – some of which can be difficult for students to make sense of...

In this lesson, students contrast multiplication and addition of surds to understand how they are different but related.

THE DIFFICULTY

Look at these statements. Are they true or false? Why?

$\sqrt{12} + \sqrt{3} = \sqrt{15}$ $\quad\sqrt{12} - \sqrt{3} = \sqrt{9}$

$\sqrt{12} \times \sqrt{3} = \sqrt{36}$ $\quad\sqrt{12} \div \sqrt{3} = \sqrt{4}$

Students may need calculators to be sure. Three of the right-hand sides are square roots of perfect squares, so students may recognise these integers ($\sqrt{36} = 6$, $\sqrt{9} = 3$, $\sqrt{4} = 2$). The multiplication and the division are correct, but the addition and the subtraction aren't.

THE SOLUTION

How can we be sure whether these are true or false?

We can show that $\sqrt{12} \times \sqrt{3}$ must equal $\sqrt{36}$, because ***squaring*** each of these expressions gives us the same value:

$$(\sqrt{12}\sqrt{3})(\sqrt{12}\sqrt{3}) \stackrel{?}{=} \sqrt{36}\sqrt{36}$$
$$(\sqrt{12}\sqrt{12})(\sqrt{3}\sqrt{3}) \stackrel{?}{=} \sqrt{36}\sqrt{36}$$
$$12 \times 3 \stackrel{?}{=} 36$$

After squaring, the left-hand side equals the right-hand side, meaning that $\sqrt{12} \times \sqrt{3} = \sqrt{36}$.

If we try this with the ***addition***, we get a problem:

$$\sqrt{12} + \sqrt{3} \stackrel{?}{=} \sqrt{15}$$
$$(\sqrt{12} + \sqrt{3})^2 \stackrel{?}{=} \sqrt{15}\sqrt{15}$$
$$(\sqrt{12})^2 + 2\sqrt{12}\sqrt{3} + (\sqrt{3})^2 \stackrel{?}{=} \sqrt{15}\sqrt{15}$$
$$12 + 2\sqrt{12}\sqrt{3} + 3 \stackrel{?}{=} 15$$

So, we can see not only that these are ***not equal*** but that the left-hand side in a case like this is ***always*** going to be bigger (because of the extra $2\sqrt{12}\sqrt{3}$ term, which ***must be positive***).

So, $\sqrt{12} + \sqrt{3} > \sqrt{12 + 3}$, ***not*** $\sqrt{12} + \sqrt{3} = \sqrt{12 + 3}$.

Square rooting is ***sub-additive***, which means that, unless either a or b is zero, $\sqrt{a + b} < \sqrt{a} + \sqrt{b}$. The radical symbol $\sqrt{}$ does ***not*** behave like multiplication (e.g., something like $3(a + b) \equiv 3a + 3b$). Square rooting is ***not distributive over addition***, like multiplication is.

However, we ***can*** simplify $\sqrt{12} + \sqrt{3}$, by using what we have seen about ***multiplication*** of surds. The number 12 has a ***square factor*** (4), and so we can write $\sqrt{12} = \sqrt{4}\sqrt{3}$, and, because 4 is a square number, $\sqrt{12}$ is equal to $2\sqrt{3}$.

So, $\sqrt{12} + \sqrt{3} = 2\sqrt{3} + \sqrt{3} = 3\sqrt{3}$.

(This last step is just 'counting in $\sqrt{3}$s' and is analogous to collecting like terms.)

Students should check this on their calculator.

We can simplify additions and subtractions of surds in this kind of way whenever there is a square number that is a factor of the number being square rooted.

What would $\sqrt{12} - \sqrt{3}$ be equal to?

This time, $\sqrt{12} - \sqrt{3} = 2\sqrt{3} - \sqrt{3} = \sqrt{3}$. It may look strange to write $\sqrt{12} - \sqrt{3} = \sqrt{3}$ (students may think it should be $\sqrt{6} - \sqrt{3} = \sqrt{3}$), but it is correct.

Checking for understanding

To assess students' understanding, ask them to find as many surd expressions as they can that are equal to $3\sqrt{5}$. For example, they could start with $4\sqrt{5} - \sqrt{5}$ and convert this to $\sqrt{80} - \sqrt{5}$. There are many possibilities, and generating lots of these is an excellent way to practise using these ideas. They could try to make the $3\sqrt{5}$ as concealed as possible and to make some that look as though they ***might*** be equal to $3\sqrt{5}$ ***but aren't!***

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