Lesson plan: MATHS KS3

BOXING CLEVER

Solving a problem that requires integer solutions provides an opportunity to develop students' algebra, says **Colin Foster**

In this lesson, students have to find a solution to a problem where the unknowns must be non-negative integers. This constraint means that the initial problem, which may appear to be quite open, in fact has a unique solution. Students have the opportunity to consider related Diophantine equations and determine systematically whether they have one solution, more than one solution or no solutions. Arriving at a point where they are sure about this allows students to see the power of mathematical thinking.

STARTER ACTIVITY

Q *Look at this puzzle. Try it in pairs.*

A shop sells two sizes of boxes.
Small boxes cost £5.
Large boxes cost £7.
I bought some boxes from the shop and spent a total of £41.
How many boxes of each size did I buy?

Students will probably not immediately see how to approach this problem. It may look as though there is insufficient information – the problem is ***under-specified*** – and there might be many possible combinations of small and large boxes that could work. In fact, that is not the case. If students are stuck, encourage them to try some numbers and see what happens.

MAIN ACTIVITIES

Discuss the problem posed in the starter.

Q How did you get on? What do you think the answer is?

Students may be quite unsure how to proceed, but they may be able to say that a possible solution is definitely ***wrong***, which is a start. Students might take an algebraic approach, and write something like $5x + 7y = 41$, which is a good way to begin. But, to solve simultaneous equations in two unknowns, a second equation is normally needed. We would need a statement like "I bought a total of 7 boxes", but this information is not provided.

The constraint that x and y must be non-negative integers actually contributes a lot of information to the solution of this problem. I cannot buy fractions of a box, or a negative number of boxes; the number of boxes must be zero or a positive integer. It turns out that, in this case, this actually makes the solution unique. By trial and improvement, students may stumble across the solution (4, 3); i.e., 4 small boxes and 3 large boxes: $5 \times 4 + 7 \times 3 = 41$.

Q Are there any other possible solutions? How can you tell?

Unless students are systematic about their search, they will have no idea whether (4, 3) is the only solution or just one of many and it is good to make them realise this.

Download another brilliant KS3 lesson plan on teaching algebra through puzzles at **teachwire.net/algebrapuzzles**

WHY TEACH THIS?

Many real world problems require integer solutions. This adds an interesting constraint to students' equation solving.

KEY CURRICULUM LINKS

+ Use algebra to generalise the structure of arithmetic, including to formulate mathematical relationships

+ Make and test conjectures about patterns and relationships; look for proofs or counterexamples

Q What numbers do you need to try if you want to be sure of catching all possible solutions?

This is hard, and students will need time to think about it and discuss how they might tackle it. Students might draw the graph $5x + 7y = 41$ and look for any integer lattice points in the first quadrant that the line goes through. They will need to be quite accurate to be sure that any 'hits' or 'misses' are really as they seem!

An alternative approach is to think in terms of multiples. Since 7 is bigger than 5, it is more efficient to consider the different possible multiples of 7, subtracting each from 41 to see whether a multiple of 5 remains. (This may also be preferable, since multiples of 5 are quicker to check for.)

y	41 - 7y	Multiple of 5?
0	41	No
1	34	No
2	27	No
3	20	**Yes!**
4	13	No
5	6	No
6	-1	No

Q When solving equations, what difference does it make if the solutions have to be integers?

It is clear that for values of y greater than 6, the values of $41 - 7y$ will become increasingly negative. This corresponds to the fact that more than 5 large boxes will, on their own, cost more than £41, leaving no money for any small boxes. So, the table confirms that there is only one solution, which happens when $y = 3$ and x must be $20/5 = 4$.

It will take students some time to reach a convincing explanation of why there can be only one solution to this problem. If students do get this far, ask them to explore other amounts of money besides £41. In other words, for which values of c in $5x + 7y = c$ is there exactly one solution, more than one solution, or no solutions? Keen students could try all amounts between £40 and £50.

c	Solutions (x, y)
40	(1, 5) and (8, 0)
41	(4, 3)
42	(0, 6) and (7, 1)
43	(3, 4)
44	(6, 2)
45	(2, 5) and (9, 0)
46	(5, 3)
47	(8, 1)
48	(4, 4)
49	(0, 7) and (7, 2)
50	(3, 5) and (10, 0)

DISCUSSION

You could conclude the lesson by discussing how students worked on the problem.

Q. *What did you do first? How did you tackle the problem? How did you find the solution? How did you convince yourself that there definitely weren't any other solutions? What related problems did you work on? What did you learn about how to solve problems like this?*

The integer solutions to $5x + 7y = c$ for values of c from 40 to 50 are given in the table (left).

You could tell students that problems like this, where the solutions are restricted to integers, are called ***Diophantine Equations***, after the 3rd century mathematician Diophantus of Alexandria.

ADDITIONAL RESOURCES

A related problem is available at **nrich.maths.org/595**

NEXT STEPS

Confident students could try varying the 5 and the 7 in the equation, considering the more general Diophantine equation $ax + by = c$. What happens if a and b are not coprime?

THE AUTHOR

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