Lesson plan: MATHS KS4

COMBINING SQUARE NUMBERS

Square numbers provide an interesting context for students to make conjectures and use algebra to prove them, says Colin Foster

DOWNLOAD
a FREE task sheet containing empty number grids to accompany this lesson plan
teachwire.net/square-numbers

WHY TEACH THIS?
In this lesson, students try combining square numbers in various ways and examine and explain the patterns that they find.

KEY CURRICULUM LINKS
- Make and test conjectures about the generalisations that underlie the patterns and relationships; look for proofs or counter-examples; begin to use algebra to support and construct arguments
- Select appropriate concepts, methods and techniques to apply to unfamiliar and non-routine problems

In this lesson, students examine how square numbers, and sums and differences of square numbers, relate to multiples of 4. Modulus arithmetic is not used explicitly, but students are encouraged to make conjectures about what is possible and to prove these using simple algebra. There are many opportunities for surprise and for students to see the power of algebra in establishing with certainty what must always happen for all numbers, no matter how large.

Q How do the square numbers relate to multiples of 4?

STARTER ACTIVITY

Q Colour in all the square numbers (perfect squares) in the grid (Fig 1). What do you notice?

The purpose of this activity is for pupils to remind themselves what square numbers are, and to begin to observe a pattern that will lead to some important thinking later on.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16
17	18	19	20
21	22	23	24
25	26	27	28
29	30	31	32
33	34	35	36
37	38	39	40
41	42	43	44
45	46	47	48
49	50	51	52
53	54	55	56
57	58	59	60

Fig 1

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16
17	18	19	20
21	22	23	24
25	26	27	28
29	30	31	32
33	34	35	36
37	38	39	40
41	42	43	44
45	46	47	48
49	50	51	52
53	54	55	56
57	58	59	60
61	62	63	64

Fig 2

The coloured-in grid is shown on the right (Fig 2).

Pupils will notice that the only cells that are shaded are in the leftmost and the rightmost columns, never the middle two, and that they alternate left-right-left-right... They may also notice that the square numbers go odd-even-odd-even...

MAIN ACTIVITY

Q Why should this be? Do you think this will always happen forever?

Students will need help in working out how to answer these questions. The odd-even-odd-even… alternating pattern in the square numbers is the easiest to make sense of. Since odd × odd = odd, and even × even = even, then, since the natural numbers (1, 2, 3, …) alternate between odd and even, their squares will do the same.

To explain why all the squares are in only the first and the fourth columns, it helps to think in terms of multiples of 4. Then there are only four kinds of number:

- Multiples of 4
- 1 more than a multiple of 4
- 2 more than a multiple of 4 (the same as 2 less than a multiple of 4)
- 3 more than a multiple of 4 (the same as 1 less than a multiple of 4)

Students could use m to represent a multiple of 4, and so could write these four possibilities as $m + r$, where r is equal to:

- 0 for the multiples of 4
- 1 for the numbers which are 1 more than a multiple of 4
- 2 for the numbers which are 2 more than a multiple of 4
- 3 for the numbers which are 3 more than a multiple of 4

Now, if we square $(m + r)$, we get $(m + r)^2 = m^2 + 2mr + r^2$. In this expression, m is clearly a factor of both of the first two terms, so how the entire expression relates to the multiples of 4 depends entirely on the r^2.

The 'Conclusion' column shows that there are only two possibilities for the square numbers, and that these alternate as we go through the

Kind of number	r	r^2	Conclusion
multiple of 4	0	0	multiple of 4
1 more than a multiple of 4	1	1	1 more than a multiple of 4
2 more than a multiple of 4	2	4	multiple of 4
3 more than a multiple of 4	3	9	1 more than a multiple of 4

natural numbers, so that they appear with equal frequency. This means that we end up with half of the squares being of the form $4n$ and the other half being of the form $4n + 1$. Students could try to explain this argument to their neighbours. They might find that drawing pictures helps.

Q Now, explore what happens, in terms of multiples of 4, when you have:

- *The sum of two square numbers*
- *The difference between two square numbers*
- *The sum of three square numbers*
- *The cube numbers*
- *The powers of 4*

There is lots here for students to make conjectures about, and then try to explore and explain algebraically.

DISCUSSION

Q What did you find out? For example, can you make all the natural numbers by summing 2 square numbers? Or from the difference between 2 square numbers? If you can, can you do it more than 1 way? If not, why not?

A summary of what is possible is given in the table below. This table shows that, for example, a cube is never 2 more than a multiple of 4, or, looked at another way, never twice an odd number. The same is true for the difference of two squares.

	Relationship to multiples of 4			
Expression	$4n$	$4n + 1$	$4n + 2$	$4n + 3$
p^2	Y	Y	N	N
$p^2 + q^2$	Y	Y	Y	N
$p^2 + q^2 + r^2$	Y	Y	Y	Y
$p^2 - q^2$	Y	Y	N	Y
p^3	Y	Y	N	Y
p^4	Y	Y	N	N

ADDITIONAL RESOURCE
A very interesting related task produced by the NRICH team can be found by visiting nrich.maths.org/whatspossible

GOING DEEPER
If you add up any two square numbers and double it, when do you get the sum of two squares? Lewis Carroll, the author of *Alice in Wonderland*, proved that this always happens. Suppose that a^2 and b^2 are the two square numbers. Then $2(a^2 + b^2) \equiv (a + b)^2 + (a - b)^2$

ABOUT OUR EXPERT
Colin Foster is a Reader in Mathematics Education at the Mathematics Education Centre at Loughborough University. He has written many books and articles for mathematics teachers.
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