LESSON PLAN

GROWING SHAPES

TODAY YOU WILL...

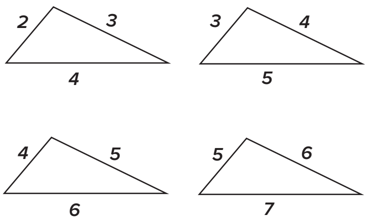
INVESTIGATE A SEQUENCE OF DIFFERENT SHAPES TO SEE WHAT YOU CAN FIND OUT ABOUT THEM

Rich mathematical tasks, with lots of potential outcomes, can present all learners with engaging and worthwhile challenges, says **Colin Foster...**

In mathematics, short, closed tasks often require a lot of careful differentiation if all learners are to benefit equally from them. On the other hand, richer, more open-ended challenges can offer everybody (perhaps even the teacher) appropriate mathematical experiences. In this lesson, learners explore a sequence of gradually larger shapes and consider what mathematical questions they can ask about them.

STARTER ACTIVITY

Q: Have a look at these shapes. They are* not *drawn accurately. What is the same and what is different about them?



There are lots of things that learners might say. All of the shapes are triangles and they all have numbers written on the sides, which are their lengths. There are no units, because this is a pure maths problem, but you could put 'cm' or something after each number if you think that that would help.

All of the triangles look exactly the same, and learners might be puzzled by the idea that they are mathematical sketches and are not intended to be accurate drawings. They need to get used to this, as 'not drawn accurately' often appears beside diagrams on examination papers!

Once learners accept that the triangles represented by these sketches are not identical, there are many similarities and differences that they might point out, such as:

- The areas are getting bigger from left to right.
- The shapes are not mathematically similar.
- The top angle in the triangle is getting bigger from left to right.
- The side lengths are always consecutive numbers
- The bottom side gets 1 bigger each time (as do the other sides).

It doesn't matter if learners say things that are wrong at this stage, as these things can be addressed in the rest of the lesson. For example, they might think that the triangles are all mathematically similar because their sides are the same amount bigger each time.

Once a number of observations have been made, you could ask:

Q: What mathematical questions can you ask about this?

It can be useful to write on the board the suggestions offered, whether you think they are 'good' ones or not. At the end, learners can select an enquiry that they want to work on. You can always add in some ideas of your own if there are particular things that you want to cover. It takes time for learners to get good at posing mathematical queries. If suggestions are not forthcoming, remind them of some of their initial observations and ask them whether they can turn any of these into questions.

Some possible examples are mentioned on the next page, but it would be good to go with whatever comes up.

+ STOP PRESS: NEW MA PRESIDENT ANNOUNCED

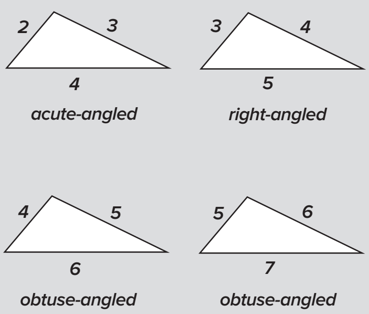
England's first teachers' subject association has appointed its new president to raise awareness of STEM in the classroom and the benefits of teaching the history of mathematics. Peter Ransom has been announced as the Mathematical Association's president 2013/2014, succeeding Oxford University professor and media personality, Marcus du Sautoy. "Mathematics teaching is experiencing a great deal of attention at the moment with the new curriculum proposals pushing for increased rigour," observes Peter. "The MA is heavily involved in discussions around both the primary and secondary curriculum. As president, I see my role as being a strong advocate for high quality mathematics teaching and a broadening of the curriculum to include a more conceptual and historical approach to the subject. Helping young people understand the significance and daily use of mathematics is essential to improving enjoyment and performance, and will ensure that the next generation is highly numerate and able to compete in the global market."
For more info on MA, visit m-a.org.uk

MAIN ACTIVITIES

1 *What do these triangles look like when drawn accurately?*

Learners could use compasses or a dynamic geometry package that they are familiar with (e.g., the free software *Geogebra* geogebra.org) to construct these triangles accurately. They could then return to the initial question about what is the same and what is different about them, now that they can look at them drawn precisely.

For example, learners might categorise them as:



(Some learners might recognise the 3-4-5 Pythagorean triple, and know that it is going to be a right-angled triangle before they draw it.)

Learners could measure the angles in each triangle and look for patterns in the answers (see below).

2 *What are the 'other' triangles in this sequence like?*

Learners might wonder where the (1, 2, 3) triangle is, that would logically come first. In fact, you cannot have a triangle with sides 1, 2 and 3, and learners who try to construct it accurately will discover why. They could explore other 'impossible triangles' and generalise what they find to the triangle identity, which says that if $a \le b \le c$ are the three sides of a (possible) triangle, then $a + b > c$. Because $1 + 2 \not> 3$, you cannot have a (1, 2, 3) triangle.

So (2, 3, 4) is the first triangle in this sequence, but is there a *last* triangle? What happens to the shapes as we continue on and on down the sequence of triangles? Will they all be obtuse-angled from now on or will we get some more acute-angled or right-angled triangles later?

HOME LEARNING

Learners could make a poster to summarise what they have found out about this triangle sequence. Alternatively, they could explore a different triangle sequence, such as where the sides are of the form $2x$, $3x$, $4x$ – multiplying rather than adding. This is much easier, because all of the triangles are mathematically similar.

3 *What are the sizes of the angles in the triangles?*

For learners who know the cosine rule, this can be a good opportunity to use it. If they use the formula on the first triangle, they can calculate any of the angles (e.g., the one at the top; call it θ) by working out

$$\cos\theta = \frac{2^2 + 3^2 - 4^2}{2 \times 2 \times 3} = -\frac{1}{4}$$

so $\theta = 104.5°$, correct to 1 decimal place.

Because each triangle has sides of consecutive integer length, it is possible to find a general formula. Writing the side lengths as x, $x + 1$ and $x + 2$, and using the cosine rule, taking θ as the angle opposite the largest $(x + 2)$ side, gives, $\cos\theta = \frac{x-3}{2x}$ for $x > 1$. (Alternatively, you could label the sides $(x – 1)$, x and $(x + 1)$.)

Some values for $\frac{x-3}{2x}$ and the angle θ are given in the table below.

x	$\frac{x-3}{2x}$	*angle θ (correct to 1 decimal place) in degrees*
2	-1/4	104.5
3	0	90
4	1/8	82.8
5	1/5	78.5
6	1/4	75.5
7	2/7	73.4
8	5/16	71.8
9	1/3	70.5
10	7/20	69.5

The change of the cosine from negative to positive results in a change in θ from greater than 90° to less than 90°.

SUMMARY

You could conclude the lesson with a plenary in which learners talk about what they have found out about the sequence of triangles. It would be interesting to probe learners' thoughts on what will happen to these shapes if they continue growing in this way forever. How will the *shape* of the triangles change as x increases?

INFO BAR

+ ADDITIONAL RESOURCES

+ THERE ARE NICE DEMONSTRATIONS OF THE TRIANGLE INEQUALITY AT: MATHSISFUN.COM/DEFINITIONS/TRIANGLE-INEQUALITY-THEOREM.HTML AND MATHWAREHOUSE.COM/GEOMETRY/TRIANGLES/TRIANGLE-INEQUALITY-THEOREM-RULE-EXPLAINED.PHP

+ STRETCH THEM FURTHER

+ LEARNERS MIGHT THINK ABOUT WHAT WILL HAPPEN TO THE ANGLE θ AS WE MOVE TO BIGGER AND BIGGER TRIANGLES. IF THEY CALCULATE, SAY, FOR A (99, 100, 101) TRIANGLE, THEY WILL FIND THAT THE ANGLE θ IS 61.0°. IF YOU IMAGINE THE TRIANGLE, YOU CAN SEE THAT IT IS GOING TO BE ALMOST EQUILATERAL. ALGEBRAICALLY,

$$\frac{(x-3)}{2x} = \frac{1-\frac{3}{x}}{2} \to \frac{1}{2}$$

AS $x \to \infty$ AND $\cos^{-1}\left(\frac{1}{2}\right) = 60$

LEARNERS COULD WORK OUT THE OTHER ANGLES IN THE TRIANGLES AND EXPLORE THEIR PATTERNS TOO. THEY COULD ALSO INVESTIGATE THE GENERAL CASE OF TRIANGLES WHOSE SIDE LENGTHS MAKE AN ARITHMETIC PROGRESSION. TAKING THE SIDES AS x, $(x + a)$ AND $(x + 2a)$ INSTEAD GIVES, BY SIMILAR WORKING, THAT

$$\cos\theta = \frac{(x-3a)}{2x}$$

BY USING HERON'S FORMULA (MATHSISFUN.COM/GEOMETRY/HERONS-FORMULA.HTML), CONFIDENT LEARNERS COULD ALSO OBTAIN AN EXPRESSION FOR THE AREA OF THE TRIANGLES.

+ ABOUT THE EXPERT

Colin Foster is a Senior Research Fellow in mathematics education in the School of Education at the University of Nottingham. He has written many books and articles for mathematics teachers (www.foster77.co.uk).