HOPPING ALONG

DECIDING WHO HAS WON A RACE SOUNDS SIMPLE, BUT IT CAN BE A BIT MORE COMPLICATED THAN IT APPEARS, SAYS COLIN FOSTER...

WHY TEACH THIS?

Measurements in everyday life are never 100% accurate. Dealing with errors in measurements is a vital skill for making sense of real-world situations.

When someone says that a distance is 50 metres, what do they mean? Measurements in real life can never be made with absolute accuracy – there is always a certain amount of error. So 50 metres could be accurate to the nearest metre, or to the nearest 10 metres, for example. Knowing within what interval the true distance lies can be very important in many applications of mathematics. When measurements are combined in a calculation, and each value has a certain amount of error, things can get complicated – and sometimes the result can be counterintuitive. In this lesson, it looks clear who has won a race until we take account of the possible errors in the distance and time measurements. Students see that in some circumstances errors can combine in interesting ways to change the entire conclusion.

STARTER ACTIVITY

*Q. Choose two of these four numbers to make the **biggest** possible answer in each of the calculations below:*

5.5   6.5   9.5   10.5

□ + □   □ − □   □ × □ $\frac{□}{□}$

*Q. Now do it again, but this time try to make the **smallest** possible answer each time.*

Give students a few minutes on their own or in pairs to try this. You might want to avoid calculators, so that students think more carefully rather than merely trying every possibility.

Q. How did you decide which numbers to use for each one?

Students will realise that for adding and multiplying the biggest two numbers are the best ones to use, so 9.5 + 10.5 = 20 and 9.5 × 10.5 = 99.75 (or with the numbers in the opposite orders) are the best solutions. But subtracting and dividing are harder. Here the largest possible answers are 10.5 − 5.5 = 5 and $\frac{10.5}{5.5} = \frac{21}{11}$, because we get a larger answer by subtracting or dividing by the ***smallest*** possible number. Students may be quite surprised by this.

To get the ***smallest*** possible answers, we need 5.5 + 6.5 = 12, 10.5 – 9.5 = 1 or 6.5 − 5.5 = 1, 5.5 × 6.5 = 35.75 and $\frac{5.5}{10.5} = \frac{11}{21}$. This time we get a smaller answer by subtracting or dividing by the ***largest*** possible number.

MAIN ACTIVITY

Give out the task sheet, available at Teachwire.net (ow.ly/4npg3p), or display it on a screen:

> Qayla and Maya go to different schools. Each school has a 50-metre running track.
>
> Qayla and Maya both like hopping races, where you hop along the track on one leg.
>
> Qayla took 82 seconds to do her race. Maya took 84 seconds to do her race.
>
> Qayla says: "I was faster than you!" Maya says: "Not necessarily!"
>
> 1. Why does Qayla think she was faster?
> 2. Why is Maya right?

Make sure that students understand what is meant by a hopping race and realise that it is much slower than running normally. Some students may be confused about the phrase "not necessarily". Maya is not saying that Qayla is ***definitely*** wrong, only that she ***might*** be.

Q. Have a think about this problem in pairs. Write down your ideas.

Some students may be confused about the fact that a larger time is slower than a smaller time: the ***faster*** person is the person with the ***shorter*** time. Once they have understood that, they will see why Qalya thinks she was faster, but they will probably be puzzled about how Maya could possibly be right. They may suggest that one of the stopwatches used for timing the girls was malfunctioning, or that someone is lying or mistaken or cheating. They may also realise that although Qayla's ***average*** speed was higher, Maya might well have begun the race travelling faster, say, and been in the lead for some or most of the race, if Qayla had a spurt at the end and hurried to the finishing line. You could clarify that when the girls say "faster" they mean "higher average speed".

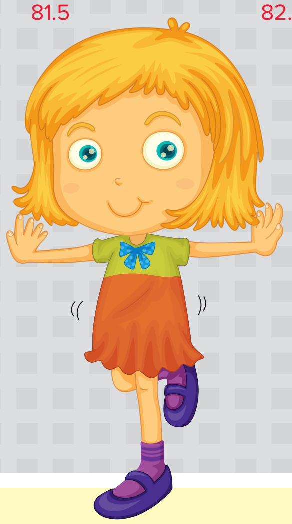
If no one mentions accuracy and rounding, then you could ask:
*Q. If Qaya's time was 82 seconds, does that mean it was **exactly** 82 seconds? What times **might** it have been?*

Students will realise that Qayla's time has been rounded to the nearest second, so it might have been anywhere in the interval:

81.5 seconds ≤ Qayla's time < 82.5 seconds

In other words, it was more than or equal to 81.5 seconds but definitely less than 82.5 seconds. On a number line this would be represented by all the times contained in the line segment:

Qayla's time 81.5 82.5 (seconds)



The coloured-in circle shows that that value is included in the interval, whereas the open circle shows that that value is not included, because 82.5 would round up to 83.

Q. Do the same thing for Maya's time and for the length of the track.

Maya's time has also been rounded to the nearest second, so it might have been anywhere in the interval:

83.5 seconds ≤ Maya's time < 84.5 seconds

Maya's time 83.5 84.5 (seconds)

The track length is a bit more difficult to do, because 50 metres might have been rounded to the nearest metre or to the nearest 10 metres. (It might even be more accurate than the nearest metre, as it is a running track, and this could be shown by writing the length as, for example, 50.0 m.) If we suppose that the track length is accurate to the nearest metre, then we have:

49.5 m ≤ track length < 50.5 m

track Length 49.5 50.5 (metres)

Now we can see that the ***fastest*** that Maya might have been is $^{50.5}/_{83.5}$ = 0.605 (correct to 3 decimal places), which is slightly faster than the ***slowest*** that Qayla might have been, which is $^{49.5}/_{82.5}$ = 0.6. (Here we have to use the idea from the starter that to get the larger answer we divide by the smaller amount, and vice versa.) So it ***is*** possible that Maya was faster than Qayla.

Students will find this hard to understand, because we have to think about the "worst case scenario" (from Qayla's point of view). Qayla appears to be the faster one, so we have to consider the ***slowest*** possible speed for her, whereas, because Maya appears to be the slower one, we have to consider the ***fastest*** possible speed for her!

Diagramatically,

track length 49.5 50.5 (metres)
Qayla's time 81.5 82.5 (seconds)
Maya's time 83.5 84.5 (seconds)
Lowest possible speed for Qayla
Highest possible speed for Maya

Q. Explain this to the person next to you.

Students will need to time make sense of this. It might help to illustrate the intervals for the speeds on a number line:

Maya's speed
Qayla's speed
0.586 ($\frac{49.5}{84.5}$) 0.6 ($\frac{49.5}{82.5}$) 0.605 ($\frac{50.5}{83.5}$) 0.620 ($\frac{50.5}{81.5}$)

(metres per second) (NOT TO SCALE)

Q. Try changing the numbers for the track length or the times. Find some situations where this can happen and some where it can't.

This is demanding, but going through the process again, with different numbers, will help students to make sense of what is happening.

DISCUSSION

You could conclude the lesson with a plenary in which the students talk about the numbers that they have tried and what they found happened.
Q. What numbers did you try? What did you find out? Did you find this "not necessarily" thing happening or not? When? Why do you think that is?

With a 50-metre track (to the nearest metre), any times 2 seconds apart from 49 and 51 seconds upwards will lead to the same effect. Working more generally in algebra is demanding, but it may be useful for the teacher to know that for a track of length d metres (correct to the nearest metre) and times t_M and t_Q seconds (with $t_M > t_Q$) (both correct to the nearest second), Qayla will not necessarily be faster than Maya if

$$\frac{d + ½}{t_M - ½} \geq \frac{d - ½}{t_Q + ½}$$

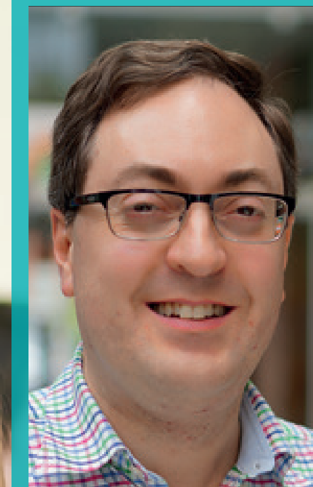
which simplifies to

$$d \leq \frac{t_M + t_Q}{2\,(t_M - t_Q - 1)}$$

In our case with $t_M - t_Q = 2$, this reduces to $t_Q \geq d - 1$, meaning that when Qayla's time is 49 seconds or more she will not necessarily be faster than Maya.

INFORMATION CORNER

ABOUT OUR EXPERT



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ADDITIONAL RESOURCES

A TASK SHEET CONTAINING THIS PROBLEM IS AVAILABLE AT TEACHWIRE.NET (OW.LY/4NPG3P)

ACKNOWLEDGEMENT

COLIN WOULD LIKE TO THANK GRANT PORTLOCK AND ANEESA AYUB FOR VERY HELPFUL DISCUSSIONS ABOUT THIS LESSON.

STRETCH THEM FURTHER

STUDENTS CONFIDENT WITH ALGEBRA COULD EXPLORE WHAT HAPPENS IN GENERAL, AS EXPLAINED ABOVE. TO MAKE THIS MORE ACCESSIBLE TO BEGIN WITH, THEY COULD FIX THE TRACK LENGTH AT 50 METRES AND THE GAP BETWEEN THE GIRLS' TIMES AT 2 SECONDS, SO THEY HAVE JUST ONE VARIABLE (QAYLA'S TIME, *t*) TO WORRY ABOUT.