WHY TEACH THIS?

Our world is full of numbers, but not all of them are created equal. Some – the prime numbers – play a critical role in enabling us to generate all the other whole numbers. Learning about the prime numbers and how they work is a crucial part of understanding about number.

PRIME SUSPECTS

SOME NUMBERS ARE DEFINITELY MORE IMPORTANT THAN OTHERS, SAYS COLIN FOSTER – AND GRASPING WHY CAN TRANSFORM STUDENTS' MATHEMATICAL UNDERSTANDING...

Sometimes students are aware of prime numbers as those whose only factors are "one and itself". This definition can lead to ambiguity about whether 1 itself should or should not be counted as prime, but another problem is that it can seem a very arbitrary definition. Why should numbers with exactly two factors be worth giving a name to and learning about? Students may learn about factorising numbers into primes without realising why this is of any more significance than factorising numbers into, say, square numbers or even numbers or any other kind of numbers. This lesson focuses on what it is about the prime numbers that makes them special – the idea that they are the building blocks of all the whole numbers greater than 1. So breaking a number down into its prime factors is a bit like breaking down a chemical molecule into is constituent elements – you really discover its structure. Since every whole number can be factorised into a unique product of primes, the prime numbers really form the basis for generating all the whole numbers that there are!

STARTER ACTIVITY

Q. Do you think that some numbers are more important than other numbers or are all numbers equally important? Why?

Students might think about this for a minute in pairs before sharing their ideas. There is no single right answer; the aim of this initial question is just for students to think about the possibility of some numbers being more important or useful than others. They might say, for example, that 12 is the most important number because it is their age, or because it has a lot of factors. They might mention telephone numbers or numbers that you see a lot in the world as being particularly important.

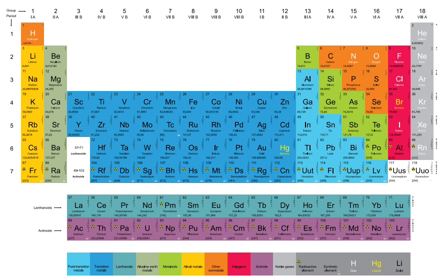
Q. I want to make all the whole numbers from 1 to 20 by multiplying numbers together. What would be a good set of starting numbers to use?

As students suggest possibilities you can introduce the idea of the ***smallest possible*** set of starting numbers that will do it. Again, this could be worth thinking about in pairs before sharing. Students might say that 2s are very useful because you can make all the even numbers, but that involves adding, and here we are restricted to multiplying only. So if all you have is a box of 2s then you can only make the powers of 2, not all the even numbers. So you need 2s and 3s, for instance, but you don't need 4s or 6s, because you can make those out of 2s and 3s. Likewise, students may think that 1s are useful, whereas in fact 1 is useful only for making 1 and doesn't contribute to making any other number, because multiplying by 1 has no effect.

The smallest set of numbers that will do it are: 1, 2, 3, 5, 7, 11, 13, 17 and 19; the first 8 prime numbers, together with 1. Students may use the language of prime numbers or they might not at this stage.

MAIN ACTIVITY

Q. Has anyone seen this before? Where? Do you know what it is?



If no one has any idea, you could suggest that they might have seen it around the science classrooms.

Q. What is special about the substances on this table? Hydrogen, helium, lithium, etc.

These are the chemical elements and every substance in the universe is made up of arrangements of just these 100 or so elements, just like every English word is made up of combinations of the 26 letters of the alphabet.

Q. Do you think there are numbers that are the building blocks of all the whole numbers? How could we find them?

These are called the ***prime numbers***. Working systematically to find them involves excluding all the numbers you can make using the primes that you have so far. A nice approach, called the ***Sieve of Eratosthenes***, is to begin with a table of numbers (see below) and:

- Colour in 1, because 1 is not a prime number.
- Leave the next empty cell, but colour in all its higher multiples.
- Keep going until you have coloured all the non-primes. (How will you know?)

1	2	3	4	5	6	7	8	9	10	11	12
13	14	15	16	17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32	33	34	35	36
37	38	39	40	41	42	43	44	45	46	47	48
49	50	51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70	71	72
73	74	75	76	77	78	79	80	81	82	83	84
85	86	87	88	89	90	91	92	93	94	95	96
97	98	99	100	101	102	103	104	105	106	107	108
109	110	111	112	113	114	115	116	117	118	119	120
121	122	123	124	125	126	127	128	129	130	131	132
133	134	135	136	137	138	139	140	141	142	143	144

A task sheet containing this table is available at teachsecondary.com/downloads/maths-resources

Students may find following these instructions difficult, so you could begin together on the board and then students could continue individually or in pairs, using a calculator where necessary.

Students are often confused about whether 1 is a prime number of not. If we think of the prime numbers as the building blocks of the integers, like the chemical elements, then 1 is not of much use (except to generate itself), so that is one way to think about why we don't include it as a prime number – multiplying by 1 doesn't do anything.

DISCUSSION

*Q. What does your table look like? Which numbers are **not** coloured in? Why? Can you really make all the other numbers (except 1) out of just these ones? How can you be sure?*

The completed table should look like this.

1	2	3	4	5	6	7	8	9	10	11	12
13	14	15	16	17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32	33	34	35	36
37	38	39	40	41	42	43	44	45	46	47	48
49	50	51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70	71	72
73	74	75	76	77	78	79	80	81	82	83	84
85	86	87	88	89	90	91	92	93	94	95	96
97	98	99	100	101	102	103	104	105	106	107	108
109	110	111	112	113	114	115	116	117	118	119	120
121	122	123	124	125	126	127	128	129	130	131	132
133	134	135	136	137	138	139	140	141	142	143	144

Continuing all the way up to 600 would give this table.

1	2	3	4	5	6	7	8	9	10	11	12
13	14	15	16	17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32	33	34	35	36
37	38	39	40	41	42	43	44	45	46	47	48
49	50	51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70	71	72
73	74	75	76	77	78	79	80	81	82	83	84
85	86	87	88	89	90	91	92	93	94	95	96
97	98	99	100	101	102	103	104	105	106	107	108
109	110	111	112	113	114	115	116	117	118	119	120
121	122	123	124	125	126	127	128	129	130	131	132
133	134	135	136	137	138	139	140	141	142	143	144
145	146	147	148	149	150	151	152	153	154	155	156
157	158	159	160	161	162	163	164	165	166	167	168
169	170	171	172	173	174	175	176	177	178	179	180
181	182	183	184	185	186	187	188	189	190	191	192
193	194	195	196	197	198	199	200	201	202	203	204
205	206	207	208	209	210	211	212	213	214	215	216
217	218	219	220	221	222	223	224	225	226	227	228
229	230	231	232	233	234	235	236	237	238	239	240
241	242	243	244	245	246	247	248	249	250	251	252
253	254	255	256	257	258	259	260	261	262	263	264
265	266	267	268	269	270	271	272	273	274	275	276
277	278	279	280	281	282	283	284	285	286	287	288
289	290	291	292	293	294	295	296	297	298	299	300
301	302	303	304	305	306	307	308	309	310	311	312
313	314	315	316	317	318	319	320	321	322	323	324
325	326	327	328	329	330	331	332	333	334	335	336
337	338	339	340	341	342	343	344	345	346	347	348
349	350	351	352	353	354	355	356	357	358	359	360
361	362	363	364	365	366	367	368	369	370	371	372
373	374	375	376	377	378	379	380	381	382	383	384
385	386	387	388	389	390	391	392	393	394	395	396
397	398	399	400	401	402	403	404	405	406	407	408
409	410	411	412	413	414	415	416	417	418	419	420
421	422	423	424	425	426	427	428	429	430	431	432
433	434	435	436	437	438	439	440	441	442	443	444
445	446	447	448	449	450	451	452	453	454	455	456
457	458	459	460	461	462	463	464	465	466	467	468
469	470	471	472	473	474	475	476	477	478	479	480
481	482	483	484	485	486	487	488	489	490	491	492
493	494	495	496	497	498	499	500	501	502	503	504
505	506	507	508	509	510	511	512	513	514	515	516
517	518	519	520	521	522	523	524	525	526	527	528
529	530	531	532	533	534	535	536	537	538	539	540
541	542	543	544	545	546	547	548	549	550	551	552
553	554	555	556	557	558	559	560	561	562	563	564
565	566	567	568	569	570	571	572	573	574	575	576
577	578	579	580	581	582	583	584	585	586	587	588
589	590	591	592	593	594	595	596	597	598	599	600

A sheet containing these completed tables is also available at teachsecondary.com/downloads/maths-resources

Q. What patterns do you notice in the table?

There are lots of things that students can point out. Students could be asked to make a poster to illustrate what they find. For example, some columns never contain any primes after the top row. Why should this be? Students might think that the 3rd column, for instance, has no primes because it consists of the multiples of 3, but that is not quite right – in general, the *r*th column contains the numbers $12n + r$, where n is the row number (beginning with row zero on the top), so if r has a common factor with 12 that is greater than 1 then there will be no primes in that column. So we have no primes (after the zeroth row) in columns 2, 3, 4, 6, 8, 9, 10 and 12. So after the zeroth row all of the primes come in columns 1, 5, 7 and 11 only.

*Q. Do you think that the prime numbers go on for ever, or do you think they eventually stop, so there is a **last** prime number?*

This can be a nice opportunity for a simple ***proof by contradiction***, although you don't necessarily need to use that language. Suppose there are exactly 1000 primes and no more, and you write them down on a piece of paper. Imagine multiplying them all together and adding 1. It would make a very big number, but would that number be prime? None of your 1000 primes would go into it, because they all leave a remainder of 1. So either this big number is a prime or it has a prime factor that is bigger than any of your 1000 primes. Either way, the 1000 primes on your list can't be all of the primes. This argument would work no matter how many primes you had written down on your list. So the primes must go on forever.

Students could begin working out which primes they need to multiply together to make each number; e.g., $18 = 2 \times 3 \times 3$ or 2×3^2. They will find that each whole number greater than 1 has ***exactly one way*** of being expressed as a product of primes (ignoring the order in which you write the product). This is called ***unique factorisation***.

INFORMATION CORNER

ABOUT OUR EXPERT

Colin Foster is an Assistant Professor in mathematics education in the School of Education at the University of Nottingham. He has written many books and articles for mathematics teachers (see www.foster77.co.uk).

STRETCH THEM FURTHER

STUDENTS COULD POSE THEIR OWN QUESTIONS ABOUT THE COMPLETED GRID. FOR EXAMPLE:

- WHAT IS THE LONGEST SEQUENCE OF CONSECUTIVE COMPOSITE (NON-PRIME) NUMBERS?
- DO THE PRIMES GET MORE SPREAD OUT AS YOU GO ON?

TWIN PRIMES ARE PRIME NUMBERS THAT ARE 2 APART (E.G., 5 AND 7). HOW MANY TWIN PRIMES CAN YOU FIND? (NO ONE KNOWS HOW MANY TWIN PRIMES THERE ARE, OR WHETHER THERE ARE INFINITELY MANY, EVEN THOUGH LOTS OF PEOPLE HAVE TRIED VERY HARD TO WORK IT OUT!)

ADDITIONAL RESOURCES

A LOVELY VISUAL ANIMATION OF THE PRIMES IS AVAILABLE AT DATAPOINTED.NET/VISUALIZATIONS/MATH/FACTORIZATION/ANIMATED-DIAGRAMS/