# THE USUAL SUSPECTS?

**Colin Foster** looks at how algebra can be taught via an engaging activity based around detective work...

Sometimes, an apparently simple mathematical scenario can lead to interesting and challenging detective work for students. Such tasks provide opportunities to draw on and develop students' emerging skills in algebra while at the same time putting the student in the driving seat. Instead of providing ready-made methods to students, in which their role is to imitate what the teacher does as accurately as they can, investigative tasks put the responsibility on the student to make sense of a potentially quite complicated situation. What can they work out? How can they find the order and patterns within the details that make a situation understandable and predictable?

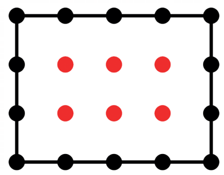
How can they use their creativity and ingenuity to make sense of a situation and represent the ideas by using and applying mathematics that they already know? And how can they communicate their ideas clearly, both to the teacher, and to one another?

## Connecting the dots

A good task for this kind of work is 'Dots in a rectangle':

*How many dots are contained inside a 4 × 3 rectangle? We* ***aren't*** *counting the dots on the boundary.*

Give students dotty squared paper and ask them to draw a 4 × 3 rectangle:



Some students may be initially confused when they are drawing it about whether they should be counting the *dots* or the spaces, because a 4 cm by 3 cm rectangle, when drawn on dotty squared paper, consists of 5 (i.e., 4 + 1) columns of dots, with 4 (i.e., 3 + 1) dots in each column. So, it is important to check at the start that all of the students have correctly drawn a 4 × 3 rectangle, as above. They should find that it contains 6 dots in the interior (shown in red here), since we're not including dots on the boundary. Make sure everyone agrees about this before proceeding.

*Can you find a connection between the size (i.e., the dimensions) of the rectangle and the number of dots contained inside it?*

This is a big question, and so students will need some time to make sense of it and come up with ideas. Some students will progress more quickly than others, and some may benefit from drawing multiple examples to help them see what's going on. They will benefit from being systematic and trying, say, 1 × 3, 2 × 3, 3 × 3, etc. rectangles, and tabulating their results clearly. (It is worth discouraging students from wasting time by drawing enormous rectangles that fill the paper. They will gain more insight from looking at a few smaller ones instead.) Other students may recognise fairly quickly that the number of rows is always going to be 1 less than the height of the rectangle, and the number of columns will always be 1 less than the width of the rectangle.

**"A mass of drawings and numbers, with suitable detective work, reduces to a (relatively) simple rule or two"**

Students might express this in words, or perhaps diagrammatically, via a generic example:

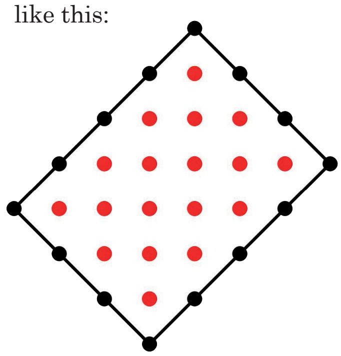
A 4 × 3 rectangle will have
−1 ↓ ↓ −1
a 3 × 2 array of dots inside.

This may be formalised algebraically by letting $x$ be the width and $y$ the height of the rectangle, and writing number of dots inside = $(x - 1)(y - 1)$.

Depending on how they think about it, other students might arrive at equivalent expressions to this one, such as $xy - x - y + 1$ or $x(y - 1) - y + 1$, and it would be a good exercise for students to verify that these are equivalent. Just because one student's answer looks different from another's, it doesn't mean that one of them is wrong! Different detectives, chasing down different leads, may arrive at the same suspect, just viewed from a different angle!

## A twist in the tale

Once students have completed this initial stage, it is then time to give everything a twist! What if we now think of a 4 × 3 rectangle a bit differently, like this:



This time, we are drawing the 4 × 3 rectangle on the same dotty squared grid, but this time it is drawn at a 45° angle to the sides of the paper. Students will see that there are now a lot more dots inside this rectangle than there were inside the previous one!

We can call this a 'gradient 1' rectangle, because the '4' side is going


up with a gradient of 1, whereas the rectangle we had before can now be thought of as a 'gradient 0' rectangle. (Note that the '4' side here is no longer exactly 4 cm long, but we can still refer to it as '4' in our new, longer diagonal units.)

Encourage students to find *efficient* ways to count the interior dots (i.e., not haphazardly, one by one!). Here, they may perceive 4 rows of 3 and 3 rows of 2, or 3 rows of 4 and 2 rows of 3, or other patterns. Counting efficiently is not only faster, but also much more reliable. It is also more likely to generate insights into the structure of the problem that will help with generalising what's going on! However the students count the interior dots, this should come to $4 \times 3 + 3 \times 2 = 12 + 6 = 18$.

*Can you find a connection between the size (dimensions) of the 'gradient 1' rectangle and the number of dots inside it?*

This is much more challenging than before, and students will need to explore the patterns in some systematically-varying 'gradient 1' rectangles and investigate the numbers that they obtain very carefully. Tracking down this second suspect is going to require more time and cunning!

## Case closed

In general, a 'gradient 1' rectangle, with dimensions $x \times y$, will contain a total of $2xy - (x + y) + 1$ interior dots. In the case above, where $x = 4$ and $y = 3$, this correctly gives us $2 \times 4 \times 3 - (4 + 3) + 1 = 18$ interior dots. One way to obtain this general formula is to exploit the structure of the interior dots mentioned above. An $x \times y$ 'gradient 1' rectangle will always contain $x$ columns with $y$ dots in each, plus $x - 1$ columns with $y - 1$ dots in each. The total number of interior dots is, therefore, $xy + (x - 1)(y - 1)$, which simplifies to the expression $2xy - (x + y) + 1$ given above.

Keen student detectives may pursue this even further, and discover that, in general, a 'gradient $m$' rectangle, with dimensions $x \times y$, will contain a total of $(m^2 + 1)xy - (x + y) + 1$ interior dots. Students who get as far as this have really come to grips with the entire 2D situation, which is quite an achievement! Extending the scenario even further, to 3 dimensions, looking at the number of dots inside $x \times y \times z$ cuboids, poised at various angles, is guaranteed to provide a considerable challenge for any student!

Investigative work of this kind gives students the chance to apply their algebraic skills, such as simplifying expressions to show that two different-looking expressions are actually equivalent. More importantly, such tasks let students draw on their mathematical powers to make sense of seemingly complicated situations. A mass of drawings and numbers, with suitable detective work, reduces to a (relatively) simple rule or two. This really shows the power of mathematics to produce generalised statements that capture a huge amount of detail, and the multiple possibilities within a concise, accurate and simple statement.

**ABOUT THE AUTHOR**

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