USING COHERENT REPRESENTATIONS OF NUMBER IN THE SCHOOL MATHEMATICS CURRICULUM

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Commercial school mathematics curricula are typically assembled in a piecemeal fashion, with responsibility for different content areas or ages distributed among a loose team of authors. These authors, like teachers planning their lessons, are likely to ask themselves the narrow question, 'What is the best way to teach this bit of content?' And this may reduce to 'What is the quickest/easiest way to achieve these particular learning objectives?' While this approach may deliver good or even excellent *individual* lesson sequences, it would seem unlikely to lead to an overall coherent curriculum, by which I mean one in which the mathematical ideas connect for the student into an intelligible mathematical story (in the sense of Dietiker, 2013). In particular, using a wide variety of different models and representations in an opportunistic, pragmatic fashion, according to the whims of the designers, and based on how well they might seem to fit each particular content area, seems to risk surrendering representational coherence across the curriculum.

Different representations of number, such as number lines, bar models, rectangular area models, circular 'pizza' models for fractions, and so on, all have the potential to offer students different perspectives, and all of these perspectives are likely to have value. However, 'the more the better' would seem to be a problematic heuristic, and introducing multiple representations in a haphazard fashion seems unlikely to be optimal (Ainsworth, 2006). In our current curriculum design work here at Loughborough University, directed towards developing a complete, free, fully-resourced set of curriculum materials (Foster, Francome, Hewitt & Shore, 2021), we have established a deliberate design principle to seek *coherence* in our use of representations. Our current thinking is that we intend to work towards this by prioritising a *single* representation of number—the number line—and exploring how far we can proceed building a deep knowledge of number through that one representation. This may seem a curious and perhaps questionable decision, and in this article I set out the rationale for our approach and give examples of how we are seeking to put this into practice in our ongoing design work.

Multiple representations and models of number

It can be helpful to distinguish 'representations' from 'models' (see, *e.g.*, Duval, 2008). We might regard the number line as a *representation* of number, because points on the line (or vectors from the origin to a point) can be used to visualise different numbers. But, the same number line representation can support multiple *models of number operations.* For example, a calculation such as 10 – 8 could be modelled on the number line using vectors starting at zero (*i.e.,* by adding the two vectors +10 and –8) or, alternatively, as 'difference', by identifying the points corresponding to 10 and 8 and discerning that they are 2 apart. These would be different models of subtraction that use the same number line representation. This distinction would be difficult to maintain through this article, however, because 'bar models', for instance, despite their name, would be representations, rather than models. So here, although I focus throughout on representations, sometimes it is more natural to use the word 'model'.

Some mathematical representations, such as the number line and Cartesian graphs, are part of the content that students must be taught, and these are non-negotiable within a given, prescribed curriculum. In contrast, other representations are purely didactic, and are used pragmatically by teachers, as and when appropriate, and do not constitute an end in themselves. They might be employed by some teachers and not by others, or with some students and not with others, or for a limited period of time, before students 'move on' from them. Common examples would include the circular 'pizza' model for fractions, bar models and double number lines. Not every professional mathematician, especially if educated in a different country, would necessarily know or have encountered all of these representations—they are optional, and so their didactical use needs to be argued for on a case-by-case basis, in terms of costs and benefits.

Rau (2017) coined the term *representation dilemma* to refer to the problem that for students to be able to make use of a representation in order to learn some mathematics they need a certain level of familiarity with that representation. This suggests that the *benefits* of any new representation that is introduced must be weighed against the *costs*. All representations have the potential to contribute to students' understanding, but, for this to be effective, students will need to commit time and cognitive space to gaining familiarity, and some degree of fluency, with that representation. Every new representation brings an opportunity cost: the time and energy could instead be spent on deepening knowledge of a previously-encountered representation (Rau & Matthews, 2017).

This argument has led us to consider whether some representations of number may have higher utility than others, and indeed whether some (or perhaps many) commonly-used representations in mathematics might profitably be dispensed with. For example, it has been remarked that students can do many more useful things with a rectangular area model than they can with a circular 'pizza' model (McCourt, 2009). While both the circular model and the rectangular model can easily represent a single fraction, such as $\frac{2}{3}$, a rectangle can be cut in perpendicular directions to show a product, such as $\frac{1}{5} \times \frac{2}{3} = \frac{2}{15}$ (Figure 1a), whereas this is challenging to represent in an illuminating way using a circular model (Figure 1b). Areas are hard to calculate or estimate if students divide circles into non-sectors (*e.g.,* Figure 2), and the relative areas of bars are generally easier to apprehend than those of sectors (Figure 3).

Consequently, might it be productive to *reduce* the total number of different representations of number presented to students, in order to focus their attention on *higher-leverage* representations? Dispensing with circular models of fractions, for instance, would allow more time for a deeper focus on other models. In addition, higher-leverage representations, such as the number line, may also be more likely to be 'canonical' representations of number, that are part of the prescribed curriculum, and so need to be taught anyway. If so, then focusing on their use, rather than being an additional cost, can be construed as an opportunity to consolidate and embed previously-learned content.

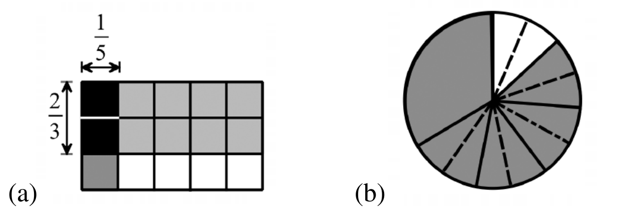

Figure 1. *Demonstrating $\frac{1}{5} \times \frac{2}{3} = \frac{2}{15}$ using (a) a rectangular area model and (b) a circular model.*

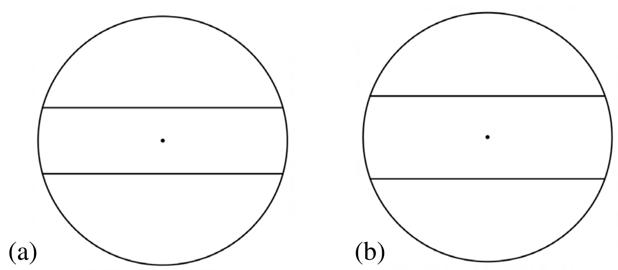

Figure 2. *Parallel lines dividing into thirds (a) the area; (b) the vertical diameter.*

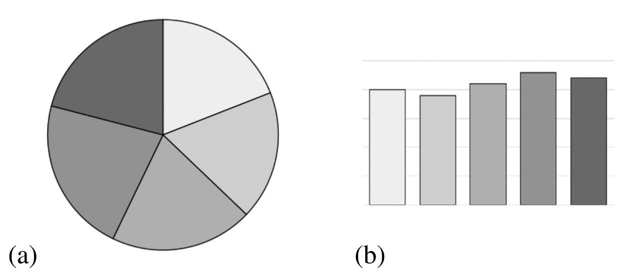

Figure 3. *The same values displayed as (a) a pie chart; (b) a bar chart.*

Enthusiasm for visual representations to support relational understanding and sensemaking in mathematics is well founded, and these may be particularly beneficial for students who are disadvantaged by traditional approaches to teaching mathematics (Gates, 2018). However, a generally positive view of diagrams may lead designers to be less cautious than perhaps we should be about allowing a proliferation of different representations, particularly when some may, on closer examination, embody problematic features.

1-dimensional and 2-dimensional representations of number

There are many ways to characterise different representations of number, such as iconic/symbolic, discrete/continuous and concrete/abstract. Here, I distinguish 1-dimensional and 2-dimensional representations of number. By 1-dimensional, so-called 'linear' models, I mean principally number lines, but I include in this category any representation that has just *one* variable or dimension, even if it is not drawn in a straight line. So, a *circular* number line, like an analogue clock or a dial on a speedometer, is still 1-dimensional, as is a spiral number line. The circular 'pizza' model for fractions, discussed above, is essentially 1-dimensional, even though its sectors occupy 2-dimensional space, because it allows only one variable to be represented (*i.e.,* angle, arc length or sector area). It is isomorphic to a finite 0-to-1 number line that has been bent around into a circle. A discrete *number track,* such as the winding squares on a snakes-and-ladders board, is also unidimensional. Movement is possible on such a board only in either a forwards or a backwards direction (ignoring the snakes and ladders themselves).

All of these models are 'linear', even though they of course have to take up 2-dimensional space in order to be visible, because their second dimension is redundant and arbitrary. For example, a 'bar model' is linear, since the dimension perpendicular to the direction of the bar can be any convenient length, and does not correspond to any relevant mathematical feature or 'represent' anything (Figure 4). In contrast, truly 2-dimensional representations of number, usually called 'area models', can represent the same number by differently-shaped but equal *areas*. For example, 6 could be a 1×6 rectangle or a 2×3 rectangle (Figure 5).

However, there is a problematic feature of these 2-dimensional representations of number. In 1-dimensional models, *all* of the relevant numbers are represented by 1-dimensional line segments and distances, but with 2-dimensional models it is *not* the case that all of the relevant numbers are represented by 2-dimensional areas. In order to show, for example, the product $2 \times 3 = 6$, we have to consider 2 and 3 to be represented by the 1-dimensional side lengths of the rectangle (Figure 6a). This distinction is particularly salient in a calculation such as $1 \times 6 = 6$ (Figure 6b), where one of the 6s is a *1-dimensional length*, whereas the other is a 2-dimensional area, and yet they represent *the same number 6*. In every case, 2-dimensional models inevitably become *mixed*-dimensional.

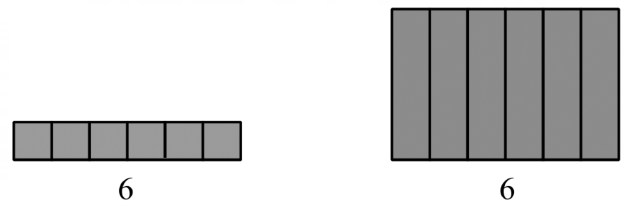

Figure 4. *Two 1D bar model representations of 6.*

Figure 5. *Different 2D representations of the number 6.*

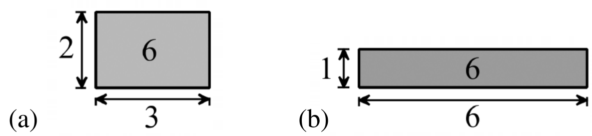

Figure 6. *The mixed-dimensional feature of area models: (a) $2 \times 3 = 6$; (b) $1 \times 6 = 6$.*

To illustrate some of the problems of mixed dimensionality, consider the multiplication of fractions. It may have been noted, back in Figure 1, that the relative linear dimensions of $1/3$ and $1/5$ were treated as the same, so the inequality $1/3 > 1/5$ was not apparent from comparison of the lengths in that figure. We have to say "$2/3$ *of the vertical height*, multiplied by $1/5$ *of the horizontal length,* is equal to $2/15$ *of the total area*". Each fraction in the calculation is a fraction of a *different* unit. This could be addressed by replacing the oblong with a unit square (as in Figure 7), so that the $2/3$ and the $1/5$ are both fractions of the *same* length. However, even if this is done, the product, $2/15$, is still a fraction of something else, and is of a very different character. The $2/15$ intuitively appears to be considerably larger than either of the numbers $2/3$ or $1/5$ which were multiplied together, since the $2/15$ area takes up considerably more space on the page than either of the line segments for $2/3$ or $1/5$. It seems likely that this unavoidable feature of the model could contribute to students' difficulty in seeing that $2/15 < 2/3$ and $2/15 < 1/5$. We still have to say "$2/3$ of one thing, multiplied by $1/5$ of the same thing, is equal to $2/15$ *of something else*".

These potential confusions persist as the quantities multiplied become increasingly abstract, for example with 'algebra tiles', as embodied in diagrams (or physical or virtual manipulatives) like those shown in Figure 8. The area model is based on a correspondence between area and number, and so it is perfectly reasonable for the same number 6 to be represented both by a 2×3 rectangle (Figure 8a) and a 1×6 rectangle (Figure 8b), since either rectangle could be broken up and fitted completely into the space occupied by the other (Figure 9a). However, it does *not* seem reasonable in Figure 8b to have the same number 6 represented *both* by the (1-dimensional) *line segment* shown in bold at the top right *and* by the 2-dimensional area at the bottom (Figure 9b).

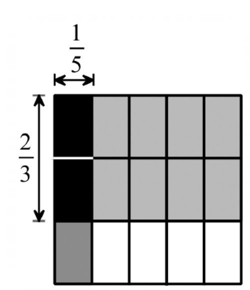

Figure 7. *A unit-square model for fraction multiplication.*

This *mixed-dimensional* feature of area models of number is problematic, because a rectangle and a line segment are not just *different things* (like two different rectangles)—they are different *kinds* of things. It may be didactically sensible to represent a number sometimes by a 1-dimensional length and other times by a 2-dimensional area—and perhaps sometimes also by other things—but to do *both* of these, simultaneously, in the same diagram, seems potentially highly confusing. It is important to note that this is not an *occasional* difficulty with 2-dimensional representations of number, in certain awkward cases—it happens every time. It would seem that one of the most basic requirements of a good representation should be that the same thing (*e.g.*, the number 6) should be represented by the same thing (*e.g., either* a line segment of length 6 *or* a rectangle of area 6, but not both at the same time).

Students might not often be heard objecting to this problem, or even perhaps be aware of it. However, it seems likely that having such a contradiction fundamentally built into the model may act as a barrier to sense making. As students progress with algebra, for example, they will encounter situations in which an expression like $3x + 6$ in Figure 8a, which is represented as 'area + area', needs to be further multiplied by some other expression. In Figure 8b, for this expression to be multiplied by $x + 1$, within this 2D model, students have to shift the quantity down a dimension, and reconceive of $3x + 6$ as 'length + length', so that they can then multiply it by

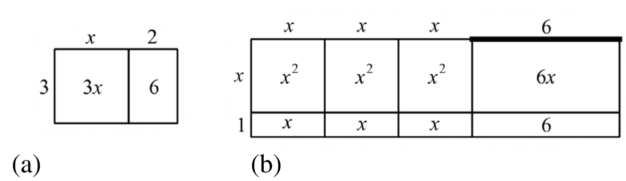

Figure 8. *Algebra tiles representing (a) $3(x + 2) \equiv 3x + 6$ and (b) $(3x + 6)(x + 1) \equiv 3x^2 + 9x + 6$.*

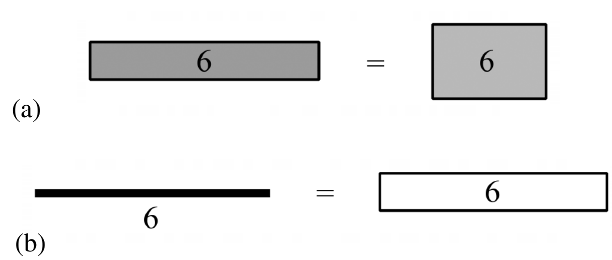

Figure 9. *(a) a reasonable equality; (b) an unreasonable equality.*

another ‘length’ (the $x + 1$), so as to obtain a quadratic expression, which is now represented as an area. This seems a potentially seriously problematic feature of the rectangular area model.

Prioritising the number line

My concerns set out above about a proliferation of multiple contrasting representations of number, together with more specific concerns about the limitations of particular models, such as the circular ‘pizza’ model, and the built-in potential contradictions of a rectangular area model, have led my colleagues and me to seek to prioritise the number line as the overarching representation of number within our curriculum design. Freudenthal (1999) called number lines a “device beyond praise” (p. 101), and in many countries number lines are familiar objects throughout primary and secondary education. Saxe, Diakow and Gearhart (2013) designed lesson sequences for teaching fractions that used the number line as the principal representational context, and Sidney, Thompson and Rivera (2019) found that number lines were better than area models for learning fraction division.

However, when moving to *multiplication and division,* rectangular area models usually predominate, and, for the reasons given above, we wish to avoid this transition, and remain true to number lines throughout. Below, I will set out how number lines could remain as the primary representation of number, even as students move to multiplicative/proportional reasoning. We think that the development of a number line into a *pair* of mutually perpendicular number lines, comprising Cartesian axes, offers a potentially more coherent approach to thinking about multiplication that avoids the dimensional problems I have outlined with area models. Although perpendicular Cartesian axes define points in *the plane*, rather than on a line, this can still be viewed as a *1-dimensional* representation of number, since, as I will show, in every case, a number is always and only represented as a *length of a line segment,* and never as an area. So, the Cartesian representation does not suffer from the mixed-dimensional problems outlined above.

All perpetually-useful models of multiplication need to be able to take students beyond ‘repeated addition’ to a continuous ‘stretch’ understanding (see, *e.g.,* Lunney Borden, Throop-Robinson, Carter & Prosper, 2021). The rectangular area model does this by allowing a continuous length to represent both the multiplier and the multiplicand, and allowing either or both of these to be non-integer. However, it does this at the cost of making the *product* 2-dimensional, leading to the problems I have discussed above. We can avoid this difficulty with 1-dimensional number lines, and the most obvious approach might be to use *parallel*, double-number lines, aligned at zero (Figure 10). These can either have the same scale (in which case mapping arrows can be useful, Figure 10a) or different scales (Figure 10b).

It could be a natural progression for students to move from a single number line to a pair of parallel number lines. However, since Cartesian graphs need to be learned anyway, we think that using *perpendicular* number lines may make this a more efficient step (Figure 11). The central challenge with working multiplicatively is to contrast multiplicative and additive thinking. Double number lines may not necessarily

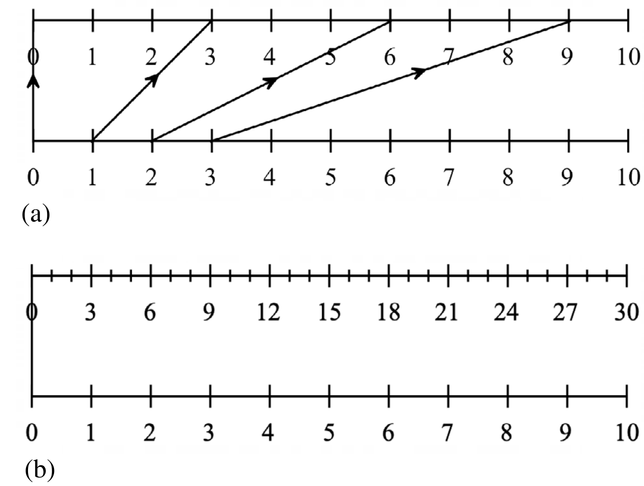

Figure 10. Double number lines representing multiplication by 3 with (a) the same scale; (b) different scales.

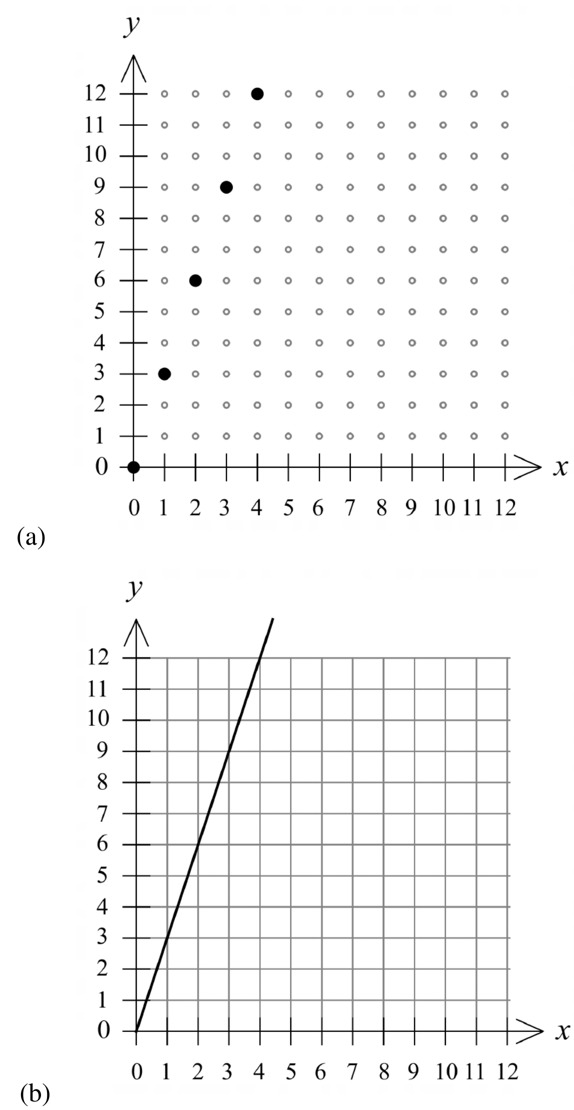

Figure 11. Cartesian graphs representing multiplication by 3: (a) discrete (integer); (b) continuous (real).

do this as effectively, since there is limited structural disincentive for students to visualise incorrect, additive results, rather than the correct, multiplicative result. For example, consider the 'ratio' question below:

> To make a drink, 5 litres of soda is mixed with 4 litres of orange juice.
>
> To make a drink that tastes the same, how many litres of soda must be mixed with 6 litres of orange juice?

In the double number line shown in Figure 12a, students experience little structural impediment to writing the *incorrect* answer 5 + 2 = 7 litres, which comes from additive thinking, compared with writing the correct answer, $5 \times 6/4 = 7.5$ litres. The structure of the double number lines fails to offer much of a barrier to this crucial error. However, in Figure 12b, with the Cartesian graph (even, arguably, if hand-drawn and imprecise), students may be more likely to appreciate that a line through the origin would have to *bend* to accommodate the incorrect value (the dashed curve in Figure 12b). The anchoring of the line through the origin (0, 0) in the Cartesian representation could have greater potential to support the need for a *multiplicative* response.

Possible objections to this approach

I now consider three possible objections to this proposed approach.

1. Prioritising any one representation over others is bad

Lee and Lee (2022) expressed well the view that:

> Because each type of model has its own affordances and constraints, heavy reliance on only one is problematic as it does not support students' construction of strong and flexible understanding of fraction concepts but is rather likely to restrict students' thinking. (p. 6)

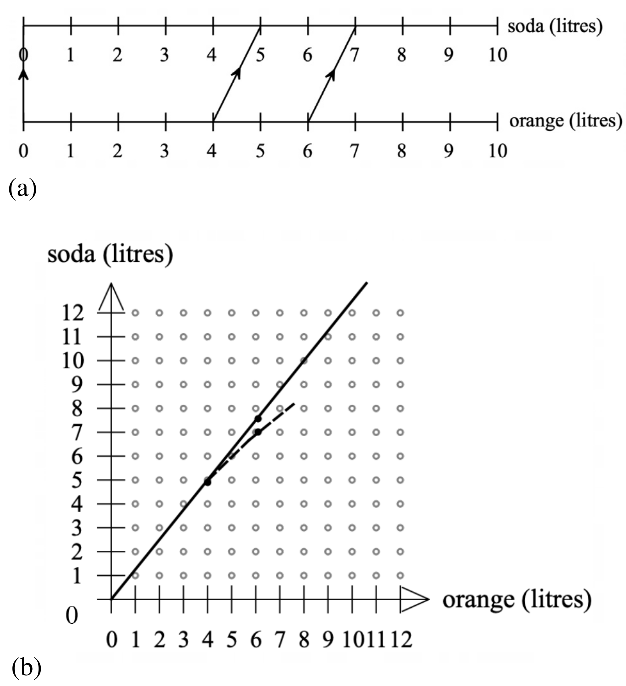

Figure 12. Incorrect (additive) solutions using (a) a double number line, and (b) a Cartesian graph.

Over-stressing one representation is considered to be detrimental, whereas converting between different models builds connections, since no single representation could ever be sufficient to embody all aspects of a mathematical concept—any representation would leave out something important (Duval, 2008). On this account, all representations have something to offer. For example, even the circular model for fractions, criticised above, has some benefits over other models, such as the way in which it is possible, when presented with a single sector, to recover 'the whole circle'. This cannot be done from a rectangular piece taken out of a larger unit square, since any $m \times n$ rectangle could be viewed as a fraction of *any* square that has sides equal to a multiple of the least common multiple of m and n. Related to this is the view that privileging one form of representation is inherently inequitable, because different students will learn in different ways and so benefit from different representations.

These would seem to be valid concerns, but the *representation dilemma* (Rau, 2017) forces us to weigh up the advantages of multiple representations against the consequent reduction in time and attention given to any individual one. It is possible that multiple representations, rather than supporting and reinforcing one another, as intended, could result in an overall less powerful picture for the student than might be obtained with one carefully-chosen representation used repeatedly and consistently. The choice between 'more is more' and 'less is more' does not seem easy to resolve by argument, and empirical research is needed to discover the benefits and drawbacks of each approach.

2. Cartesian graphs are too abstract or difficult

Seeking curricular coherence means consciously moving away from necessarily prioritising the quickest, easiest, short-term solution to teaching each individual piece of content. This means being prepared to do *more difficult* things, if they seem to have the potential to lead to greater coherence in the long run. The intention is that the investment of time and energy in an intensive focus on one primary representation (the number line), rather than multiple representations, provides the time for constantly revisiting it, and viewing it in different ways, and this could help students to build a deeper understanding. Cartesian graphs must be taught anyway in upper primary and lower secondary school, so nothing additional is being proposed, only a deeper focus on this at the expense of alternatives.

It is also important to note that using 'Cartesian graphs' does not necessarily imply the premature use of algebraic letters. Cartesian graphs can be used initially purely numerically, with a focus on structure, before any symbolic algebra is brought in. To this end, a teacher might choose at the start to avoid the language of 'x-axis' and 'y-axis', and even 'gradient' or 'slope'. The fundamental idea is of a position in 2-dimensional space being referenced by two values, one horizontal and one vertical, and this is readily experienced by students in classroom 'people math' scenarios, in which one student is given the task of tracking the position of two other students who walk back and forth along a pair of perpendicular lines.

The key idea behind all of the proposed multiplicative work is the linear $y = mx$ proportionality relationship: straight lines through the origin, with gradient m. We see $y = mx$ as fundamental to how we wish to (re)present multiplication (*e.g.,* multiplication of fractions, ratio, proportionality, speed, trigonometry, and so on).

3. Gradient is a 'rate', and so is a different kind of quantity from the *x* and *y* values

The concern here is that our model suffers from the same problem as the rectangular area model, just in a different way. In the rectangular area model, we had two linear quantities and one area. Here, we apparently have two linear quantities, x and y, and one 'rate' (*i.e.,* the gradient, m, of the line $y = mx$). However, it is straightforward to interpret m as a linear quantity simply by drawing a vertical line to the graph at $x = 1$, and this interpretation of m is perhaps simplest to work with, at least initially. For example, to introduce multiplication of fractions we might begin with multiplying a fraction by an integer, such as 3, therefore using the $y = 3x$ graph, and zooming in to non-integer values on the x-axis. If students are comfortable that, say, $3 \times 5 = 15$, then, in order to keep all of the points on the $y = 3x$ line, they need to accept that, for example, $3 \times \frac{1}{5} = \frac{3}{5}$ and $3 \times \frac{2}{5} = \frac{6}{5}$. All of this can be observed just by zooming in on the graph and requiring that non-integer points lie on the same straight line through the origin that represents multiplication by 3.

Then, by exploring how $y = mx$ looks for different integer m—*i.e.,* by moving the anchoring point $(1, m)$ up and down vertically, perhaps in dynamic geometry software—it becomes natural to accept that the *non*-integer m values will produce lines intermediate in slope between the neighbouring integer-gradient lines. So, for example, with $y = \frac{1}{5}x$, it becomes apparent that $\frac{1}{5} \times 3$ is also equal to $\frac{3}{5}$, just as $3 \times \frac{1}{5}$ was $\frac{3}{5}$ on the $y = 3x$ graph. Following this, we can see, again by zooming in, that $\frac{1}{5} \times \frac{2}{3} = \frac{2}{15}$ (Figure 13). Notice here that *all three numbers* appear as horizontal or vertical *lengths* (shown as the three bold line segments in Figure 13).

A further advantage over the rectangular area model is that the issue of commutativity is not fudged, as it can be with the rectangular area model (*i.e.,* 'just rotate the rectangle by 90°'). Here, instead, we have to consider $\frac{1}{5} \times \frac{2}{3}$ by using the $y = \frac{1}{5}x$ graph and $\frac{2}{3} \times \frac{1}{5}$ by using $y = \frac{2}{3}x$ graph, and do some work to appreciate why the answers are equal.

We would adopt a similar approach to defining trigonometric 'ratios' via the unit circle, as lengths, rather than ratios (Hewitt, 2007), which allows angles greater than 90° to be

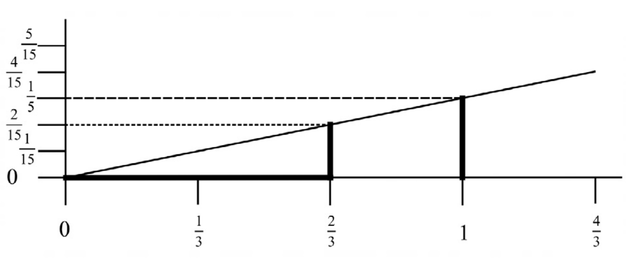

Figure 13. Using Cartesian graphs to see that $\frac{1}{5} \times \frac{2}{3} = \frac{2}{15}$.

included from the outset. We also think this approach corresponds more readily to the advanced notion of the trigonometric functions as functions of real numbers, rather than as functions of 'angles' (Foster, 2021).

Conclusion

A natural reaction might be to say that the approach proposed here seems harder than an area model approach. I do not think this is necessarily the case, but, either way, I construe the didactic design challenge not as trying to find the quickest, easiest way to address each narrow skill but to engage in the long-term investment of building the most powerfully useful and coherent models. Number lines and Cartesian graphs have enormous utility, and, if we constantly use them as our primary model, students will experience repeated engagements with these across (superficially) diverse content areas, which should be synergistic and mutually reinforcing. I see this as the essence of what might be meant by a 'coherent mathematics curriculum'. The single idea of $y = mx$—straight lines through the origin—captures much of the structure of secondary mathematics, and we revisit it again and again across different content areas.

Naturally, we do not seek to avoid 'area' as a mathematical concept. We would draw a 2 by 3 rectangle and say that its area is 6, and calculate this by multiplication. But we would expect students to think of the 6 as a different kind of object from the 2 and the 3. In elementary mathematics, an *area* of 6 is quite different from a *length* of 6, and our approach to teaching area would seek to make the dimensions of quantities highly salient, by stressing the distinction between units such as cm and cm^2. This contrasts with the rectangular area model for multiplication, which, we think, tends to obscure this distinction.

In order to emphasise proportion, we would retain a focus on $y = mx$ longer than is typical in curricula with which we are familiar, meaning that we might avoid introducing an additive constant (a '$+ c$', to make $y = mx + c$) until perhaps Grade 8. At that point, rather than contrasting 'proportional' with 'non-proportional' situations, we might instead stress the idea that when you have a non-zero c in $y = mx + c$ it merely means that you have chosen the 'wrong' origin or baseline. So, rather than saying that in $y = mx + c$ we have y *not* proportional to x, we might instead look for what *is* proportional. In this case, by transforming to $y - c = mx$, we can say that the transformed quantity $y - c$ is proportional to x. This is perhaps a more fruitful way to proceed. Likewise, in other mathematical situations in early secondary school mathematics in which there is non-proportionality, we might *seek* proportion; for example, the area of a circle is *not* proportional to its radius, but we can say that the area of a circle *is* proportional to the *square* of its radius. Similarly, *inverse* proportionality means 'proportional to the reciprocal', rather than *lack* of proportionality. This seems to be an approach that could better highlight mathematical structure through recognising multiplicative relationships as being at the heart of lower secondary mathematics. In every multiplicative situation, we are always looking for some way to draw a straight line through the origin.

Acknowledgement

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Note

[1] Manipulatives, like Cuisenaire rods or cubes, do not fit this classification, because students can perform 'linear' or 'non-linear' actions with them. For example, Cuisenaire rods might be used in a 1-dimensional line to represent numbers, or they might be used to make flat rectangles with an area of, say, 12. So their dimensionality depends on how they are used.

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