Tom Francome and Colin Foster find that simple addition is not as simple as it might seem.

When can things be added together?

The simplest of all numerical operations is surely addition. Imagine a car is waiting at a red light and another car pulls up behind. We have 'added' a second car, and have two queueing cars, which we can represent by writing 1 + 1 = 2. What could be simpler? But we think addition is more complex than it might seem at first. In particular, when can things be added together, and when can't they?

Teachers often say that things can be added only if they are in 'the same units'. So, you can add one car and another car, but if you want to add a car and a bus, then you will have to convert them into some common unit, like 'vehicles'. With fractions we can add $\frac{4}{5}$ and $\frac{7}{5}$ to get $\frac{11}{5}$, but we can't add $\frac{4}{5}$ and $\frac{2}{3}$ without finding a common denominator (i.e. a common name) and making them both counting numbers of that same unit.

But is this true? We can certainly write $\frac{4}{5} + \frac{2}{3}$, and isn't that 'adding the two fractions together'? If a and b are numbers, then we can always write their sum as $a + b$, and this is well-defined, whatever numbers a and b are. We don't have to worry about whether a and b might be expressed as fractions with different denominators. So, it seems that we can *add* any two fractions, just by writing an addition symbol in between them! The issue is that for a *simplified answer*, as a single fraction, then we need to find a common denominator. But that seems to be about making our final answer look simpler, not about whether they 'can be added' or not. So, if learners say 'You can't add fractions if they have different denominators' (at least 'without first making the denominators the same') is this correct or incorrect?

Process or product?

Posing a question for learners by writing '$\frac{4}{5} + \frac{2}{3} = \ldots$' may be considered to be misusing of the equals sign. Equals does not mean 'Work this out', but expresses a relationship of equality. It would be perfectly correct mathematically for a learner to complete this prompt by writing, for example, $\frac{4}{5} + \frac{2}{3} = \frac{4}{5} + \frac{2}{3}$ or $\frac{4}{5} + \frac{2}{3} = \frac{2}{3} + \frac{4}{5}$, showing understanding of identity and commutativity, respectively. Conventionally, when we say 'Work out $\frac{4}{5} + \frac{2}{3}$', we mean that the learner should find a *single*, simplified fraction that is equal to this sum. But this doesn't mean that the two fractions have not been added until this simplified expression has been found.

The issue here is that the expression $\frac{4}{5} + \frac{2}{3}$ represents both the process of *addition* and also the *concept of sum*. Gray and Tall (1994, p. 121) used the term *procept* to encompass both the '*process* which produces a mathematical *object*, and a *symbol* used to represent either process or object'. If learners (or teachers) suggest that $\frac{4}{5} + \frac{2}{3}$ can't be added until they have the same denominators, then that is similar to thinking of $a + b$ as a process that can't be carried out until the values of a and b are known. Tall et al. (1999, p. 4) interpreted this lack of closure as 'focus on the procedure of evaluation rather than on an algebraic expression as a manipulable procept'.

We might ask whether this means that any two quantities can be added? Surely not. It makes no sense to add a centimetre to a kilogram, because they are dimensionally different. But does this mean that addition is meaningful only when the units are the same? Certainly not. You can add a centimetre to an inch, or a pound to a kilogram, but to interpret your answer you might wish to convert to a common unit. However, this is not always helpful. What is 2 years plus 3 days? The answer is most conveniently stated as '2 years 3 days'—identical to the question! It certainly would not be 'simpler' to convert to a common unit, say 733 days, or 63 331 200 seconds, to use the standard SI unit.

It is much 'simpler' to say '2 years 3 days'. It is meaningful to add things, provided that they have the same units, or have units with the same dimensions, that *can* be converted into the same units, even if you don't convert.

It can be helpful to think of the curriculum not in terms of the content that we wish to teach, but in terms of the awarenesses that we wish to educate. Hewitt (2004) outlined a sequence of experiences attempting to educate the special awarenesses needed to add fractions. This is quite different from offering atoms (Foster, 2025), as it can be achieved in a variety of ways 'from scratch'—not in small steps, but in giant leaps. One of these special awarenesses is the need for a common name to add. Gattegno described an activity using apples and pears. We know that 2 apples and 3 apples is 5 apples, and that 2 pears and 3 pears is 5 pears. The awareness of 2 + 3 = 5 transcends the units, and can lead to learners being able to handle cm, hundreds, $2x + 3x$, 2 flinkertyflu-ths + 3 flinkertyflu-ths, 2 sevenths + 3 sevenths, and so on. Hewitt argued that learners may not need to know anything about fractions to do $\frac{2}{7} + \frac{3}{7}$, as this is an awareness about *addition*, and becomes a linguistic game with the structure '2 things plus 3 things is 5 things', though of course learning about fractions is also important!

Returning to the fruit, 2 apples and 3 pears certainly does give 5 'somethings', and this can raise the need for a common name, such as '5 fruits'. While this can be helpful for raising awareness of the need for a common name for simplifying, it cannot be stretched too far. For example, $\frac{2}{9} + \frac{3}{7}$ is not straightforwardly '5 somethings', in the same way that $2a + 3b$ cannot be simplified to 5 anything. In particular, $2a + 3b \neq 5ab$, and it has been suggested that in algebra we should make a fuss about the letters being *numbers* and not *objects*. However, they are also, in a sense, mental objects that can be manipulated (Leversha, 2010). While we might want to avoid 'fruit salad algebra' early on, perhaps it is helpful later to reinterpret $2a + 3b$ as an addition that cannot be simplified due to the different 'units'?

Teachers sometimes say that in mathematics (unlike in science) we never actually add *measures*, only *numbers*. So, we never add 1 cm and 1 cm; instead, we add 1 and 1, and interpret our answer to be 2 cm. So, we shouldn't write 'Distance d = 2 cm'. We should instead say that the distance d is the *number of centimetres* (i.e. a pure number), and so $d = 2$ means that the distance is 2 cm. Mathematically, we could then certainly add a number of kilograms (say 3) and a distance in centimetres (say 2), if we

Figure 1. Sign in New Cuyama, California, USA. (photo: Mike Gogulski, CC BY 2.5, `https://commons.wikimedia.org/w/index.php?curid=2513523`)

wanted to, but the answer, while a perfectly reasonable number (5), wouldn't correspond to anything useful in the real world. It wouldn't be 5 'anythings'. This can lead to nonsense problems, such as the one shown in Figure 1.

In a sense, units can be thought of as multiplications. For example, 2 cm can be thought of as 2 multiplied by the agreed length of a centimetre. Addition then accumulates units, so 2 cm + 3 cm = (2+3) cm. This follows the distributive law: $ac + bc = (a + b)c$. So, 2 cm + 3 cm seems to have a lot in common with $2x + 3x = (2 + 3)x$. Multiplying and dividing things with different units is not a problem if we can make sense of it in the real world (Foster, 2019).

So, when can you add things together? Addition is a well-defined mathematical operation, and it seems addition can always be expressed symbolically. Whether the result can be simplified or has any meaning depends on the context, and in particular the units.

References

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