THE TRIANGLE:
HALF A RECTANGLE OR HALF A PARALLELOGRAM?

By **Tom Francome**

I have often heard teachers debating whether to think of a triangle as half a rectangle or half a parallelogram. For example, "Every triangle is half a parallelogram, but few are half of a rectangle. Indeed, the majority of triangles are not right-angled" (Pearce, cited in McGrane and McCourt, 2020, p.287). However, this is usually because they are thinking of triangles with sides parallel to the grid on which the triangle is drawn or the edges of the page. The arguments tend to go something like this. It is easy to *see* that a right-angled triangle is half a rectangle, there are two copies of the same triangle. If the base is parallel to the grid or the edge of the page, you can add an altitude to create two right angled triangles and then, with a little effort, reason that each of these is half its respective rectangle, so the original triangle is half the rectangle with the same base (Figure 1).

If the triangle is sheared, as in Figure 2, so one angle at this horizontal base is obtuse, it is then harder to *see* this as half the rectangle. I think this is the kind of situation teachers are thinking of as it is easier to imagine the triangle as being half of the parallelogram with the same base – again, the diagonal of the parallelogram makes two congruent triangles. Two of the possible parallelograms for the triangle in Figure 2 are shown in Figure 3. This depends on which side's midpoint you rotate the triangle around to create the parallelogram (for more detail see Francome, 2021).

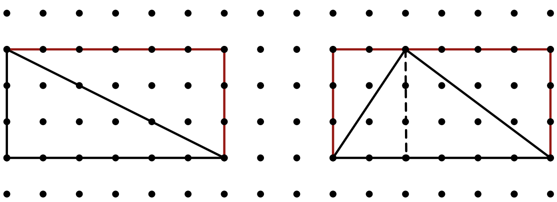

Fig. 1: The triangle is half the rectangle – easily seen when the chosen base is parallel to the edge of the page.

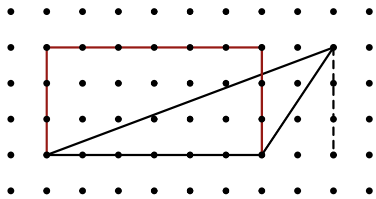

Fig. 2: If the triangle has an obtuse angle on the base parallel to the grid, it can be harder to see it as half the rectangle with the same base.

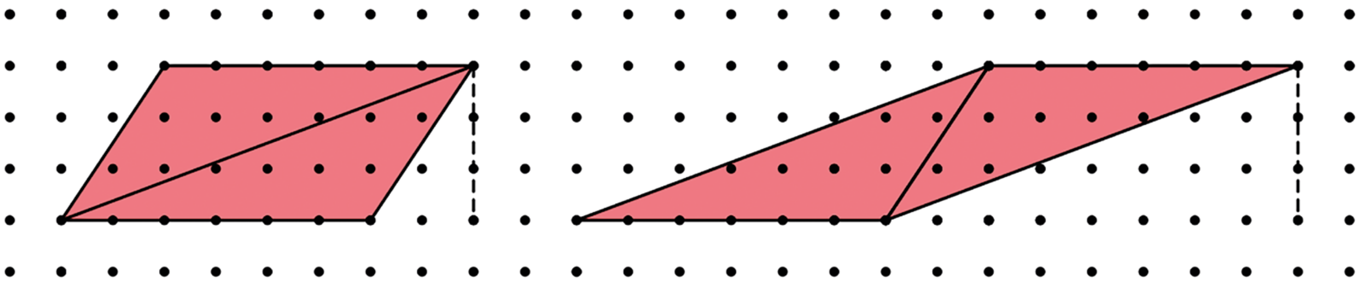

Fig. 3: In this image, it is easier to see the triangle as being half of each of the two parallelograms.

Some extra triangles could be added and then subtracted from a larger rectangle as in Figure 4, but that argument can feel a bit more numerical than geometric – the triangle doesn't feel obviously half the rectangle with the same base. It's not always possible to split the rectangle into two congruent copies of the triangle but it is possible to split a parallelogram into two congruent halves. This is perhaps what people mean when they say a triangle isn't half a rectangle.

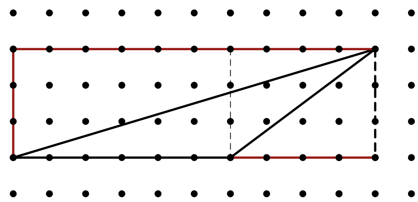

Fig. 4: Surrounding an obtuse angled triangle with a rectangle. Helpful for calculating the area, but less helpful for seeing the area as half the rectangle with the same base?

Of course, that is unless you choose a different side to be the base. If you do that, there is ***never*** a situation where you cannot easily see the triangle as half a rectangle with the same base. Every triangle, whether acute-angled, right-angled, or obtuse-angled can be seen as one of the two cases in Figure 1 above. A rectangle can be drawn on the longest side of every triangle and the triangle can easily be seen as half that rectangle.

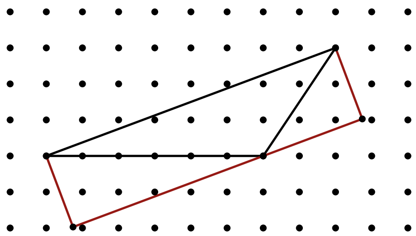

Fig. 5: Any side can be the base so the triangle is always half the rectangle.

Considering this situation got me wondering.

If you construct a triangle on a square grid, then can you always 'construct' a rectangle that the triangle is half of from each of the three bases?

The rule for this kind of grid construction (Note 1) is that you may only draw straight lines between lattice points or intersections of lines. I suggest you explore this before reading on. A square grid is an ideal space to explore geometry, the dots are evenly spaced apart, with each dot being one unit away from the next one, both horizontally and vertically. Using a square grid is beneficial because it makes it simple to figure out if lengths and angles are equal or different (See Francome, 2021). Parallel and perpendicular lines can be easily constructed by drawing dot-to-dot copies of line segments either parallel or rotated. I have chosen the example in Figure 5 so the missing vertices of the rectangles are not lattice points of the grid. How might they be constructed? One possible hint can be found in Figure 7.

Now I am not suggesting that this necessarily makes finding the triangle's dimensions easy, but that is not the point! Learners need to work on the *concept* of area of a triangle rather than just the procedure. Thinking about the triangle as having three possible bases and the respective perpendicular heights can help learners to think more flexibly about area and furthermore about the concept of a triangle as part of a connected web of ideas.

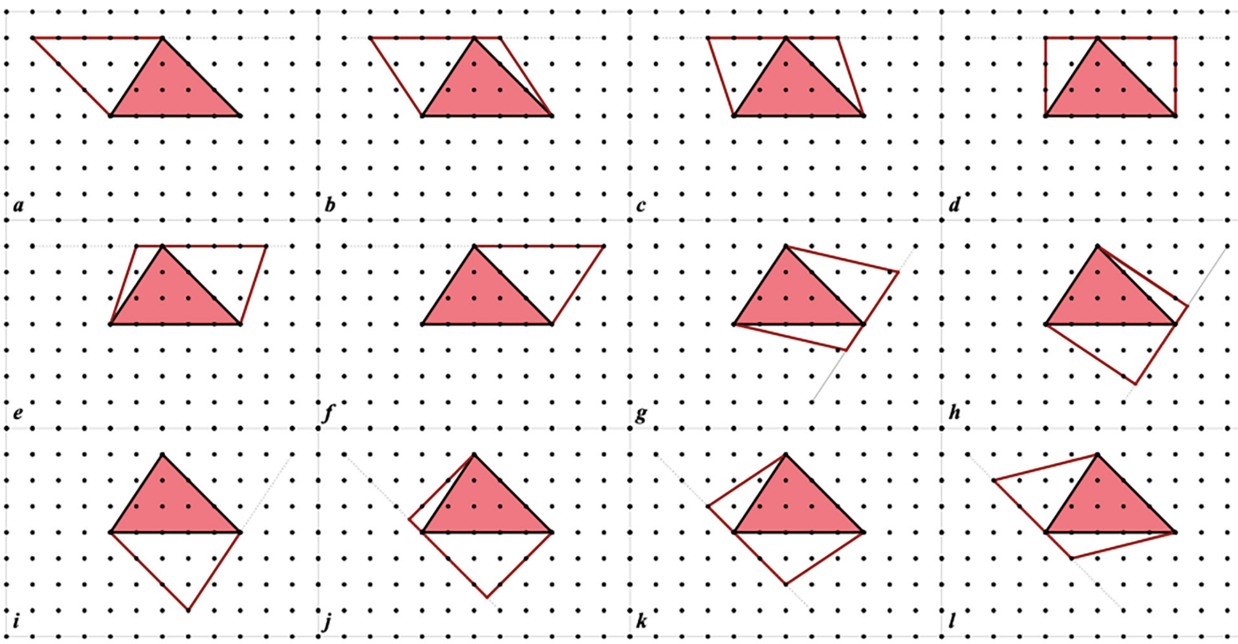

Fig. 6: Parallelograms surrounding a triangle so the triangle has a common base and half the area. The progression, a-l, can be thought of as a filmstrip or flipbook. You might try convincing yourself that each triangle is indeed half of the surrounding parallelogram and also consider how each one can be constructed using the grid.

Within this connected web, learners can come to see a triangle as both half a rectangle and as half a parallelogram. An image I find helpful is shearing surrounding parallelograms parallel to a base of the triangle. A filmstrip such as in Figure 6 can be used to draw attention to *every* side of the triangle as a potential base of a rectangle/parallelogram with the same perpendicular height as the triangle. That rectangle can be sheared through various parallelograms into each of the others whilst preserving the area. Shearing can help learners think geometrically because it's so visual (Shore et al., 2023). The parallelograms all have the same area, the triangle is always half, and *sometimes* that parallelogram is a rectangle.

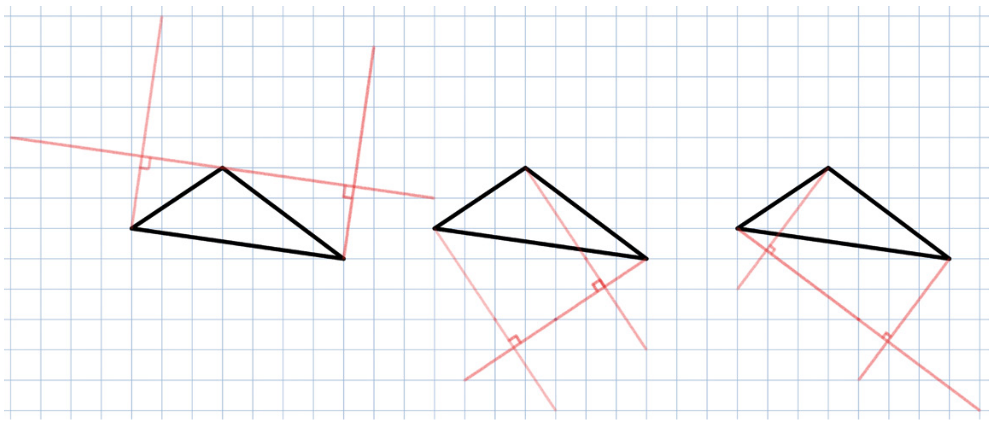

Fig. 7: Constructing the three possible rectangles the triangle is half of by choosing one side as the base and then drawing four line segments of the same length using the grid (two parallel through the apex, and one perpendicular to each end of the 'base'). The triangles can be sheared to be 'obviously' half the rectangle.

Triangles are always half a parallelogram with the same perpendicular height, whichever side you choose as the base. Triangles are also half a rectangle, whichever side you choose as the base. Furthermore, there is a neat 'construction' to obtain these rectangles when the triangle is drawn on a square grid (see Figure 7). Teachers say things like half a rectangle/parallelogram because they want learners to make sense of the concepts but thinking about shearing can help deepen that sensemaking (Shore *et al*., 2023) (Note 2). So, not only is it always possible to see triangles both ways but it is also desirable to see triangles both as half a rectangle *and* half a parallelogram.

Notes

I feel like this 'grid geometry' is almost like a different kind of geometry, (a bit like taxi-cab geometry) in terms of constructions; e.g., for parallel lines you just draw lines with the same vector rather than having to do something like this: **www.mathopenref.com/constparallel.html**

The LUMEN curriculum makes use of the relationship between parallelograms and triangles but as the goal is to develop learner sensemaking, learners also reason with the relationship between rectangles and triangles.
www.lboro.ac.uk/services/lumen/curriculum/

References

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McGrane, C., and McCourt, M. 2020 *Mathematical Tasks: The Bridge Between Teaching and Learning*, John Catt Educational Ltd.

Shore, C., Francome, T., & Foster, C. 2023 'The sheer delight of shearing', *Mathematics in School*, 52, 4 (September), pp. 32–34.

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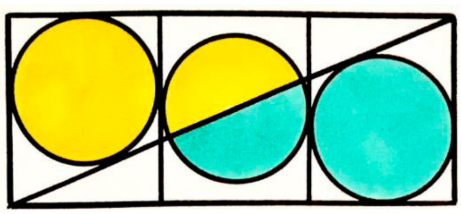
Geometry Problem 27

By Catriona Agg

The circles each have radius 1.

What's the area of the rectangle?



Send your solutions to Chris Pritchard (chrispritchard2@aol.com) and we will publish the best.