Simultaneous Equations

Cramer's rule

- If you know about matrices, find out about *Cramer's Rule*, which enables you to solve simultaneous equations such as

$$4x+3y=-2$$
$$5x+2y=1$$

  in your head!
- What happens if you have *three* unknowns?

Cramer's Rule gives the solution to

$$ax+by=p$$
$$cx+dy=q$$

as

$$x=\frac{\begin{vmatrix} p & b \\ q & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \text{ and } y=\frac{\begin{vmatrix} a & p \\ c & q \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}.$$

The denominators are the determinant of the coefficients, as laid out in the equations. The numerators have the x-coefficients replaced by the constants to evaluate x and the y-coefficients replaced by the constants to evaluate y. It is not as complicated as it looks, and is not too hard to remember.

So in this case, $x=\dfrac{\begin{vmatrix} -2 & 3 \\ 1 & 2 \end{vmatrix}}{\begin{vmatrix} 4 & 3 \\ 5 & 2 \end{vmatrix}}=\dfrac{-7}{-7}=1$ and $y=\dfrac{\begin{vmatrix} 4 & -2 \\ 5 & 1 \end{vmatrix}}{\begin{vmatrix} 4 & 3 \\ 5 & 2 \end{vmatrix}}=\dfrac{14}{-7}=-2$

The calculations can be done mentally, provided you are good at working out determinants and are allowed to stare at the equations while you think!

The rule works just as well for three variables, so the solutions to

$$ax+by+cz=p$$
$$dx+ey+fz=q$$
$$gx+hy+iz=r$$

are:

$$x=\frac{\begin{vmatrix} p & b & c \\ q & e & f \\ r & h & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}},\ y=\frac{\begin{vmatrix} a & p & c \\ d & q & f \\ g & r & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}} \text{ and } z=\frac{\begin{vmatrix} a & b & p \\ d & e & q \\ g & h & r \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}.$$

This is probably a bit too hard to do without paper unless lots of the elements are zero!

Risqué simultaneous equations

A mother is 21 years older than her child.

In exactly 6 years from now, the mother will be exactly 5 times as old as the child.

Where is the father?

Letting the mother's age be m years and the child's age be c years leads to the simultaneous equations:

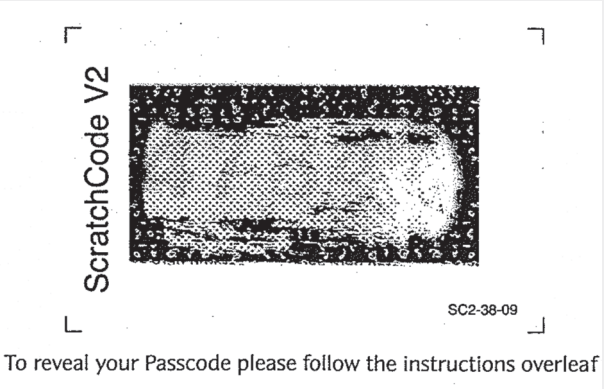
$$m = c + 21$$
$$m + 6 = 5(c + 6)$$

Substituting the first equation into the second gives

$$c + 21 + 6 = 5c + 30$$

So $-3 = 4c$, giving $c = -\frac{3}{4}$ year; i.e., –9 months. So if the child is 9 months before birth then the father must be...

My PIN – a true story ...



After scratching off the security material I can make out only the first 4 digits of my 5-digit online banking PIN code: It is 4784? (I must have scratched too hard!)

Here is what they say about using it:

- Input your 16 digit debit card number (or membership card number)
- You will then be prompted to input 2 random digits from the passcode (never disclose the full passcode to anyone - we will only ever ask you for 2 random digits)

- What is the probability that I will be asked only for digits that I know?
- If I fail to give the correct numbers three times in a row, I will have to request a new PIN number. What is the probability that that will happen?

The call centre have ${}^5C_2$ possible pairs of numbers they can ask for. Since I cannot read the fifth one, there are only ${}^4C_2$ possible pairs that I can answer, so the probability that I will be OK is $\frac{{}^4C_2}{{}^5C_2}=\frac{4!3!2!}{2!2!5!}=\frac{3}{5}$, so a better than 50% chance.

Alternatively, there are only four pairs of requested digits that will give me a problem, since there are only four other digits that the fifth digit can be paired with, so the probability of not being able to satisfy the call centre requirements is $\frac{4}{{}^5C_2}=\frac{4\times 3!2!}{5!}=\frac{2}{5}$.

A simplistic calculation leads to a probability of failing three times of $\left(\frac{2}{5}\right)^3=\frac{8}{125}=6.4\%$; a pretty small but not negligible amount. However, if I keep a record on the first (and possibly second) attempts of my failed guess for the fifth number, then I can improve on this slightly, since I obviously won't try the same fifth number twice.

The Association of Teachers of Mathematics
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