instant maths ideas

FOR KEY STAGE 3 TEACHERS

number and algebra

nelson thornes

Volume 1

Number and Algebra

Colin Foster

Introduction

Teachers are busy people, so I'll be brief.
Let me tell you what this book *isn't*.

- It *isn't* a book you have to make time to read; it's a book that will *save* you time. Take it into the classroom and use ideas from it straight away. Anything requiring preparation or equipment (e.g., photocopies, scissors, an overhead projector, etc.) begins with the word "**NEED**" in bold followed by the details.

- It *isn't* a scheme of work, and it isn't even arranged by age or pupil "level". Many of the ideas can be used equally well with pupils at different ages and stages. Instead the items are simply arranged by topic. (There is, however, an index at the back linking the "key objectives" from the *Key Stage 3 Framework* to the sections in these three volumes.) The three volumes cover ***Number and Algebra*** (1), ***Shape and Space*** (2) and ***Probability, Statistics, Numeracy and ICT*** (3).

- It *isn't* a book of exercises or worksheets. Although you're welcome to photocopy anything you wish, photocopying is expensive and very little here needs to be photocopied for pupils. Most of the material is intended to be presented by the teacher orally or on the board. Answers and comments are given on the right side of most of the pages or sometimes on separate pages as explained.

This is a book to make notes in. Cross out anything you don't like or would never use. Add in your own ideas or references to other resources. Put "8R" (for example) next to anything you use with that class if you want to remember that you've used it with them.

Some of the material in this book will be familiar to many teachers, and I'd like to thank everyone whose ideas I've included; in particular, Keith Proffitt ("17 ÷ 5", section 1.4.4) and Stephen Mack ("Insert the Signs", section 1.12.9). I'm particularly grateful to those people who have discussed some of these ideas with me; especially Keith Proffitt, Paul Andrews, John Cooper and Simon Wadsley. Special thanks go to Graham Foster for expert computer behaviour management!

Colin Foster
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Contents

Volume 1 – Number and Algebra

1.1 Place Value

- We have a strange way of writing numbers, where the same digit represents a different amount depending on where it comes in the number. When reading long numbers, pupils need to count the columns from the *right* and then read the number from the *left*. Spaces between Th and H, M and HTh, etc., are difficult for many pupils to "see".
- House prices: The value of a house (a "place" where you live) depends on
  - what it's like (number of rooms, condition, age, etc.); and
  - where it is (near to shops/schools, quiet neighbourhood, etc.).

  Two identical houses in different streets can be worth very different amounts: location matters. Similarly, a number **4** is worth very different amounts in 2**4**78.63 and 36.87**4**2.
- Why it matters: Being one column out can make a very big difference; e.g.,
  - a nurse giving an injection – is it 0.01 cm^3 or 0.001 cm^3? (10 × difference!)
  - writing a cheque – is it £340.00 or £3400.00? (10 × difference!)
- Can continue the "houses" analogy by relating place value columns to streets:

M	HTh	TTh	Th	H	T	U	t	h	th	tth
	4	0	0	0	0	0				

M is the "up-market" end of town (where the millionaires live): a number-4-house in M-street is worth ten times as much as a number-4-house in HTh street.

1.1.1 Imagine I have just one each of the digits 1, 4 and 5 (could write them on cards to emphasise only one of each).
What different numbers can I make?
How much is the 4 worth in each number?

Answers: 145, 154, 451, 415, 514, 541; not including decimals (could write them in a systematic order so we know we've got them all).
e.g., in 145, value of "4" = 4 × 10 = 40

1.1.2 Oral work:
How much is the 4 worth in these numbers?

3**4**3	**4**333	333**4**.33	33.**4**33	33.3**4**

*Answers: not just "hundreds" but "**four** hundred".*

40	*4000*	*4 or 4 units*	*0.4*	*0.04*

1.1.3 Use all the digits 1,2,3,4,5 and 6, once each to make numbers in which the 4 is worth 40 000, 4000, 400, 40, 4, 0.4, 0.04, 0.004.

*e.g., 1**4**6 532, 36**4** 215, etc.*

1.1.4 **NEED** newspaper pages containing numbers. Find numbers containing a 4 that is worth these amounts: 40 000, 4000, 400, 40, 4, 0.4, 0.04 and 0.004.

Could be money, numbers of people or anything else. It's possible to do this with all kinds of newspapers or magazines.

1.1.5 Why do we have zero? What's the point of a number that isn't worth anything?

The symbol "0" holds place differently in bases other than 10; e.g., in base 2 the number "100" is actually 4.

Answer: it's a "place holder" – it shows how much the other digits are worth.
Other cultures have used different number systems (e.g., Roman numerals have no zero).
"Zero" wasn't counted as a "number" until relatively late.

1.1.6 Popeye. Find out what place value has to do with the history of the cartoon character Popeye.

Apparently spinach isn't quite as good a source of iron as was originally thought. A decimal point in the wrong place gave a false impression in a scientific report!

1.2 Decimals

- Money can be a useful context because decimal currency is so familiar (people who can't work out 0.25 + 0.85 may know that 25p + 85p = £1.10). But decimals aren't limited to money contexts. Finding "real-life" numbers with more than 2 dp can be difficult: one example is lap times in Formula 1 car racing, which are measured in thousandths of a second.
- Number-line work is vital to see that decimals aren't necessarily "small"; they're "normal" numbers and they fill in the gaps between the integers – hence the first task (1.2.1).
- Plotting points on number-lines helps with seeing the difference between 7.1 and 7.01, and that 0.1 more than 7.9 is 8.0 and not 7.10.

1.2.1 I'm thinking of a number between 1 and 10. What could it be?

Could illustrate using an acetate of horizontal lines marked off in tenths (see sheet).

e.g., 7 – too big – 3 – too small – 5 – too small – 6 – too small [could pause here to think about what that implies] – 6.5 – [show where it is on the number-line] – too big – 6.3 – too small – etc., zooming in, homing in on, say, 6.42. (Can have the number you've chosen written on a piece of paper taped under someone's desk – to prove you didn't "cheat"!)

1.2.2 **NEED** "In-betweens" sheet.
Tell me a number in between 43.711 and 43.71.

Lots of right answers (infinitely many, in fact): Can you tell me another number that fits? Can you think of a bigger number than that that works as well?

Extension: Perhaps using a different colour pen (or pencil) fill in in-between your in-between answers (one between the low number and your number; one between your number and the high number).

Could use < and > signs to write answers down.

As there are so many correct answers, pupils could mark each other's work to save the teacher a very tedious job!

A harder task would be to find a decimal in between $\frac{6}{13}$ and $\frac{7}{15}$. Convert each to decimal, find an in-between number and convert that back to a fraction.

For example, $0.462 = \frac{231}{500}$.

1.2.3 **NEED** metre stick marked off in tenths or an acetate of horizontal lines marked off in tenths (see sheet), or sketch it on the board.
Ask questions like "If this end is 3.4 and this end is 3.5 what is this?" (point to a mark along the line)

Can make this quite tricky; e.g., one end can be 14.3 and the other end 17.6.

If the left end is a *and the right end is* b*, then each tenth of the line is worth* $\frac{1}{10}(b-a)$ *and the* n^{th} *mark along has the value* $\frac{n}{10}(b-a)+a = a(1-\frac{n}{10})+\frac{n}{10}b$.

1.2.4 Oral work: Go around the class counting up in 0.3's, say, beginning at 5.
Will we ever hit 100?
What's the closest we'll get?

No, because 100 – 5 = 95 and 95 ÷ 0.3 isn't an integer. Starting at 7 would work.
After 317 pupils we'd get to 100.1 (closest).

If Sally said 14.5 what would Mark (next pupil or a pupil some distance away) have to say? What number would George have to say for Lena to say 12.8? etc.

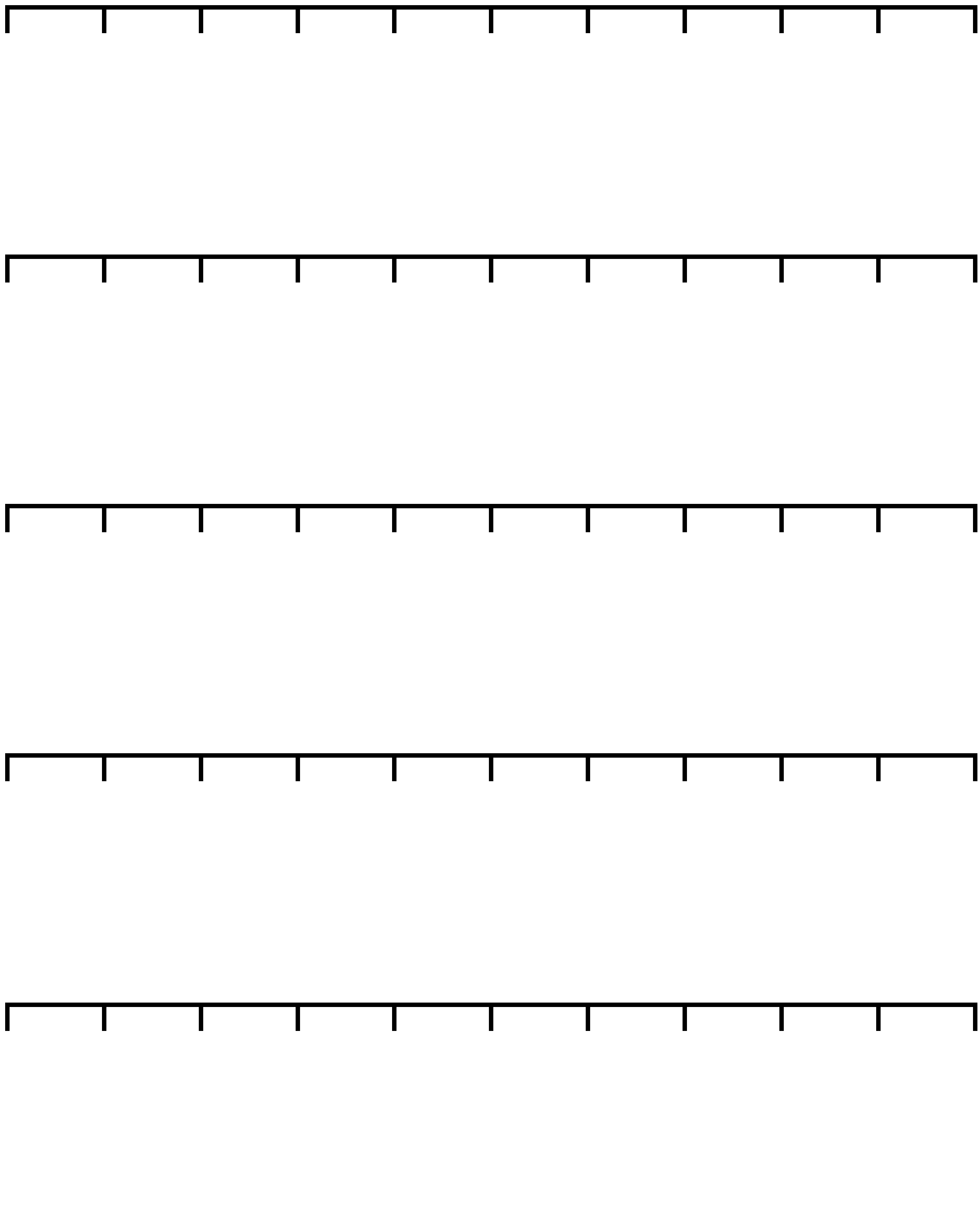
Keeping the same "add 0.3" rule or using a different one.

In-Betweens

Write in the space *any* number *in between* the two you're given.
There are lots of possibilities. The first one has been done already.

5	6	120	68	7
6.3				
7	10	130	72	8
2	14	103	6.2	4.9
3	15	104	6.4	5.1
1.1	0.6	13.6	26.8	0.81
1.6	0.9	13.8	27.2	0.83
64	3.57	2.9	1.45	79
88	3.6	3	1.46	80
32.08	2.8	0.75	29.3	0.007
32.1	2.85	0.8	29.6	0.009
3.8	1.029	15.72	30.04	3.07
3.82	1.03	15.78	30.05	3.15
2.5	7.38	0.11	0	0
2.6	7.39	0.12	0.1	0.01

Extra Task Can you make up some that are harder than these?



1.3 Multiplying and Dividing by Powers of 10

- Some pupils see this as completely different from "normal" multiplying; it isn't, it's still repeated addition. Ten of every digit moves it one more column to the left.
- It may be better to think of the **digits** all shifting rather than the decimal point leaping (although calculator displays give the opposite impression).
  So the key question is "What happens to all the digits when we multiply/divide by 10/100/1000?" "They move" "Which way?" "How far?" Need to see that × 100 is × 10 followed by × 10, etc.

1.3.1 It may be helpful to draw place value columns or use pre-printed sheets (see sheets). Diagonal arrows can show the movement of the digits to the left or the right.
e.g., one question could be 0.045 × 10
Each question has two lines: on the 1st line write 0.045 in the correct columns; then on the 2nd line write 0.450 in the correct columns – diagonal arrows show that each digit has moved 1 place to the left.

units 1's	•	tenths 0.1's	hundredths 0.01's	thousandths 0.001's
0	•	0	4	5
0	•	4	5	0

A5 sheets work well; the A4 one is suitable either for a pupil who needs more space or for putting onto an acetate.

Write questions on the board or use ones from a textbook.

You can use different coloured diagonal arrows for multiplying (shifts to the left) and dividing (shifts to the right).

Some pupils won't need to use the sheets for long; some not at all. But others will see them as a resource and ask for them again and again, or they can draw their own.

1.3.2 Use questions such as 3.4 × ? = 34000

Pupils can make up their own and check using a calculator.

Answer 10 000
This can lead in to standard form; 3.4×10^{5} *means* 3.4 × *10 "five times";* 3.4×10^{-5} *means* 3.4 ÷ *10 "five times".*

1.3.3 "To multiply by 10, add a zero onto the right-hand end of the number."
When does this rule work and when doesn't it?

Answer: Fine for integers, but not for decimals. The corresponding rule for dividing by 10 (take off a zero) works only for multiples of 10; i.e., when the answer *is an integer.*

This is an example of a rule that is appropriate in certain contexts but not in others (as most rules are).

The Decimal System

	thousands 1000's	hundreds 100's	tens 10's	units 1's	•	tenths 0.1's	hundredths 0.01's	thousandths 0.001's
1					•			
					•			
2					•			
					•			
3					•			
					•			
4					•			
					•			
5					•			
					•			
6					•			
					•			
7					•			
					•			
8					•			
					•			
9					•			
					•			

	thousands 1000's	hundreds 100's	tens 10's	units 1's	•	tenths 0.1's	hundredths 0.01's	thousandths 0.001's
1					•			
					•			
2					•			
					•			
3					•			
					•			
4					•			
					•			
5					•			
					•			
6					•			
					•			
7					•			
					•			
8					•			
					•			
9					•			
					•			

	thousands 1000's	hundreds 100's	tens 10's	units 1's	•	tenths 0.1's	hundredths 0.01's	thousandths 0.001's
1					•			
					•			
2					•			
					•			
3					•			
					•			
4					•			
					•			
5					•			
					•			
6					•			
					•			
7					•			
					•			
8					•			
					•			
9					•			
					•			

1.4 Rounding

- Rounding to the nearest integer, or nearest 10, 100, etc. and decimal places (another way of saying "nearest tenth", "nearest hundredth", etc.) need to be clear before venturing into significant figures.
- It's important to think about how accurately numbers are needed in different contexts; e.g., football crowds to the nearest 1000, country populations to the nearest 1 000 000.
- Calculators often generate long decimal answers to simple questions: e.g., working out the thickness of a 2p coin. How can I measure it accurately? Many possible approaches. One is to measure several at once. 15 coins measure 29 mm so one coin would be $29 \div 15 = 1.9333333333...$ What would be a sensible answer to give? This relates to how accurately we knew the original measurement.
- The FIX mode on calculators can be helpful or unhelpful. The simplest way to turn it off is usually to use a biro to press the "reset" button at the back of the calculator.
- The *most significant digit* (the 1st significant figure) is the one that's worth the most, and it's the first non-zero digit from the left. The 2nd and subsequent significant figures *can* be zeroes (even though they're worth nothing); they are just the next digits you come to going to the right.
  For example, in 4371, 4 is the most significant digit (put a "1" above it and write "2, 3, 4" above the next three digits – their "significance level"). To round 4371 to 1 sf means rounding to the nearest 1000 (the column that the 1st sf is in). Rounding 0.372 to 2 sf means rounding to the nearest hundredth (or 2 dp). Pupils sometimes confuse the most significant digit with the rounded answer. (4371 to 1 sf isn't just 4!) We need a number close to the number we're rounding – that's the whole point! On a number-line it's clearer that we're looking for a *close* number.
- "5 or more we round up, otherwise it stays the same" is a helpful rule. "Rounding down" can be confusing, since the digit doesn't change – it only makes sense in a number-line context.
- Though it may not be expressed, a problem is sometimes that 47, say, seems much "closer" in form or name to 40 (it's got a 4 in the tens column) than it does to 50. This leads to confusion when asked "what's 47 closer to, 40 or 50?". The questioner means "closer on a number-line".

1.4.1 **NEED** stack of cards with different integer and decimal numbers (see sheets).
Hold them up "What's this to the nearest integer?" etc.
Or use acetate and point (see other sheet).

It doesn't matter if use the same cards again because you'll probably ask for a different degree of rounding.

These cards (or the acetate) have multiple uses; e.g., "double these numbers", "find two numbers that add up to a certain amount", etc.

1.4.2 A less arbitrary/pointless task than rounding meaningless numbers is to use calculators to generate numbers to round.
Interesting contexts include square roots and converting fractions to decimals (see sheets).

Answers are below.

1.4.3 If I've rounded a number to the nearest integer and I get 18, what's the smallest/biggest it could have been?

The main difficulty is in seeing that 13.499999999… is 13.5. Can use $\geq$ and $<$ signs to clarify that the lower bound is the smallest number that will round up to the value whereas the upper bound is the smallest *number that* just won't *round down to it.*

Max and min possible values (upper and lower bounds).

Pupils are often sceptical that 0.9999… = 1, but since 0.3333… = $\frac{1}{3}$ (exactly, not approximately, so long as the 3's go on forever), you can multiply both sides by 3 and get 0.9999… = 1. (You can do a similar "trick" with 9 × $\frac{1}{9}$.)

1.4.4 **NEED** "17 ÷ 5" sheet.
Different answers to the same numerical question.

Sometimes it's best to be as accurate as possible, sometimes to round up, sometimes to round down.

1.4.5 It's useful to round the same number to different degrees of accuracy – hence a table is useful, say with ten numbers down the left side. This can be written on the board.

number	to nearest int.	to nearest 10	to nearest 100	to nearest 1000
34624.51	34625	**34620**	34600	34000

Or you can have nearest integer, 1 dp, 2 dp, etc. or 1 sf, 2 sf, 3 sf, etc.

You can't guess from the size of a number how accurately it should be rounded; the context determines that.

Must be careful each time to round the original *number and not the previous rounded answer (see 34620 in the table).*

1.4.6 The trickiest cases are, e.g., 2499.7 rounded to the nearest integer.
Can imagine rounding and carrying a 10 and then a hundred, but it may be easier to ask which two integers it's in between.

Answer: 2500 (it's in between 2499 and 2500 and nearer to 2500). Whenever rounding gets difficult, it's usually best to imagine or sketch a number-line.

Fractions, Decimals and Rounding ANSWERS

	1	2	3	4	5	6	7	8	9	10
1	1.000	2.000	3.000	4.000	5.000	6.000	7.000	8.000	9.000	10.000
2	0.500	1.000	1.500	2.000	2.500	3.000	3.500	4.000	4.500	5.000
3	0.333	0.667	1.000	1.333	1.667	2.000	2.333	2.667	3.000	3.333
4	0.250	0.500	0.750	1.000	1.250	1.500	1.750	2.000	2.250	2.500
5	0.200	0.400	0.600	0.800	1.000	1.200	1.400	1.600	1.800	2.000
6	0.167	0.333	0.500	0.667	0.833	1.000	1.167	1.333	1.500	1.667
7	0.143	0.286	0.429	0.571	0.714	0.857	1.000	1.143	1.286	1.429
8	0.125	0.250	0.375	0.500	0.625	0.750	0.875	1.000	1.125	1.250
9	0.111	0.222	0.333	0.444	0.556	0.667	0.778	0.889	1.000	1.111
10	0.100	0.200	0.300	0.400	0.500	0.600	0.700	0.800	0.900	1.000

Rounding Square Roots ANSWERS

x	$\sqrt{x}$ **(calculator display)**	**3 dp**	**2 dp**	**1 dp**	**nearest integer**
1	1.000000000	1.000	1.00	1.0	1
2	1.414213562	1.414	1.41	1.4	1
3	1.732050808	1.732	1.73	1.7	2
4	2.000000000	2.000	2.00	2.0	2
5	2.236067977	2.236	2.24	2.2	2
6	2.449489743	2.449	2.45	2.4	2
7	2.645751311	2.646	2.65	2.6	3
8	2.828427125	2.828	2.83	2.8	3
9	3.000000000	3.000	3.00	3.0	3
10	3.162277660	3.162	3.16	3.2	3
11	3.316624790	3.317	3.32	3.3	3
12	3.464101615	3.464	3.46	3.5	3
13	3.605551275	3.606	3.61	3.6	4
14	3.741657387	3.742	3.74	3.7	4
15	3.872983346	3.873	3.87	3.9	4
16	4.000000000	4.000	4.00	4.0	4
17	4.123105626	4.123	4.12	4.1	4
18	4.242640687	4.243	4.24	4.2	4
19	4.358898944	4.359	4.36	4.4	4
20	4.472135955	4.472	4.47	4.5	4

The number of answers that round to n is simply $2n$. So forty numbers round to 20.

2.43
15.84
120.072
83.8465
47.9261

30.893
15.263
0.48
93.2983
60.971

47.613 0.027

0.6

1 243

313.5

0.18

2.5650

39.82

537.2

4 291.3 0.027

Fractions, Decimals and Rounding

Look at the table below. We can use the numbers around the edge to make fractions.
Take the *horizontal* number as the **numerator** (top number of the fraction), and
the *vertical* number as the **denominator** (bottom number).
So the shaded box would be $\frac{4}{3}$ (or $1\frac{1}{3}$).

Use a calculator to convert this fraction to a decimal.
Do this by working out the *numerator* divided by the *denominator*.
So 4 ÷ 3 = 1.3333333…
Round all your answers to *3 decimal places*.
So we get 1.333

Complete the table.
Remember each time to do the *horizontal* number divided by the *vertical* number and to write all the answers to 3 decimal places.

	1	2	3	4	5	6	7	8	9	10
1										
2										
3				1.333						
4										
5										
6										
7										
8										
9										
10										

What patterns do you notice in the table? Can you explain them?

Rounding Square Roots

Use a calculator to find the square root of the number x each time.
Round your answers to 3 dp, 2 dp, 1 dp and to the nearest integer.
Round from the *original answer* each time and not from your previous rounding.

x	$\sqrt{x}$ **(as on calculator)**	**3 dp**	**2 dp**	**1 dp**	**nearest integer**
1					
2					
3					
4					
5					
6					
7					
8					
9					
10					
11					
12					
13					
14					
15					
16					
17					
18					
19					
20					

How many square roots are equal to 1 when rounded to the nearest integer?
How many round to 2?
How many round to 3?
Is there a pattern? How many do you think would round to 20?

$17 \div 5$

17 people need to go home from a party by taxi. Each taxi can hold 5 people. How many taxis do you need?

Cycles cost £5 per day to hire. You have £17. How many days can you hire a cycle for?

Grandma has £17 in her purse and she wants to share it out among her 5 grandchildren. How much will they get each?

I have 17 metres of curtain material and I need to make 5 curtains. How long will each curtain be?

A plate of cakes contains 5 cakes. If I bake 17 cakes how many plates can I fill and how many will be left over?

Floppy disks are stored in boxes that hold 5 disks. I have 17 disks to store. How many boxes will I need?

A Boeing 737 has 5 seats across the plane. 17 friends go to Aberdeen. How many rows do they need?

Write $\frac{17}{5}$ as a mixed fraction.

Work out the answers to these problems.
Are the problems really the same or different?
Make up a different question. Which one is it most like?

1.5 Decimal Calculations

- Addition and subtraction can be done using number-lines or in columns with the decimal points lined up vertically. Number-lines are particularly useful where negative numbers are involved. Sometimes "thinking money" is helpful.
- Multiplication is probably easiest by the "gelosia" (boxes) method. You can find the position of the decimal point in the answer by following the decimal points in the two numbers vertically and horizontally until they meet and then travelling down the diagonal.
- Division is sometimes easiest by writing the division as a fraction and multiplying numerator and denominator by 10 (or powers of 10) until they are both integers. Then a normal division method will work. Sometimes repeated subtraction is a simpler process.

1.5.1 Puzzle pictures (colour in the answers to produce a picture).

Often popular and available in books.

1.5.2 Cross-sum puzzles. (It's best to put the decimal points onto the answer grid.)

Pupils can make up their own.
Check answers with a calculator.

1.5.3 Box Method (*gelosia*) multiplication (see sheet).

*Answers: **1.** 28; **2.** 52; **3.** 18.5; **4.** 1312; **5.** 19.5; **6.** 561.2; **7.** 21.66; **8.** 174; **9.** 0.1746; **10.** 11.56; **11.** 22.792; **12.** 4676.8; **13.** 563.744; **14.** 1529.0854.*

1.5.4 **NEED** "Patterns in numbers" sheet and calculators.
The aim is to look for a pattern in multiplying and dividing big and small numbers.

"If you multiply by a number bigger than 1 then the answer is bigger than what you started with." "If you multiply by a number smaller than 1 then the answer is smaller than what you started with." And the opposite way round for dividing.

Ask questions like "if I multiply 17 by 0.3 what can you tell me about the answer?"

Answer: smaller than 17, because multiplying 17 by a number <1, but bigger than 0.3 because multiplying 0.3 by a number >1, etc.

Assume that a and b are both positive, and try to complete a table of inequalities like this.

		ab	$\frac{a}{b}$ and $\frac{b}{a}$
$a>1$	$b>1$	$ab>a$ $ab>b$	$\frac{a}{b}<a$ $\frac{b}{a}<b$
	$b<1$	$ab>b$ $ab<a$	$\frac{a}{b}>a$ $\frac{b}{a}<b$
$a<1$	$b>1$	$ab>a$ $ab<b$	$\frac{a}{b}<a$ $\frac{b}{a}>b$
	$b<1$	$ab<a$ $ab<b$	$\frac{a}{b}>a$ $\frac{b}{a}>b$

Easy to get confused.
Try numbers to check them.

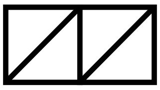
1.5.5 **NEED** "Number Investigation" sheet (2 copies on the sheet).
The aim is to explore the concept of inverses.

A number n and its reciprocal $\frac{1}{n}$ are "multiplicative inverses". Dividing by one of them is equivalent to multiplying by the other. $n\times\frac{1}{n}=1$

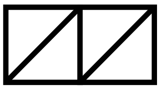
Box Method Multiplication

Use the "box method" to work out these multiplications.
Write the answers after the equals signs.

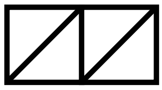
1 14 × 2 =



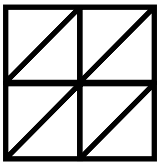
2 13 × 4 =



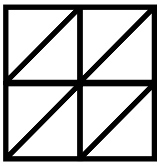
3 3.7 × 5 =



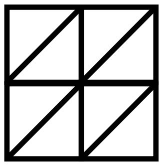
4 82 × 16 =



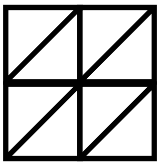
5 7.5 × 2.6 =



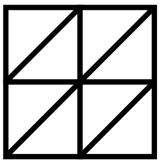
6 92 × 6.1 =



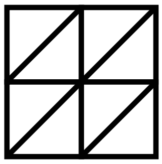
7 0.38 × 57 =



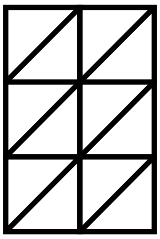
8 2.9 × 60 =



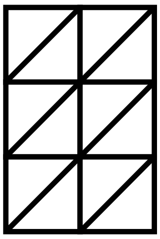
9 0.18 × 0.97 =



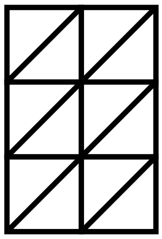
10 8.5 × 1.36 =



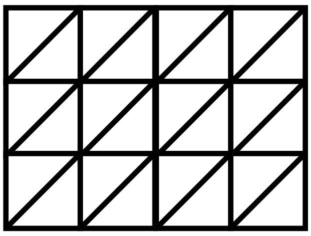
11 4.4 × 5.18 =



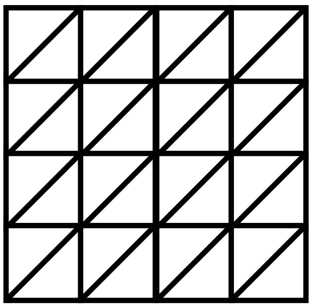
12 63.2 × 74 =
(careful!)



13 178.4 × 3.16 =



14 2.839 × 583.6 =



Patterns in Numbers

$600\,000 \div 0.002 =$

$6 \times 0.2 =$

$6 \times 2 =$

$6 \div 2 =$

$600 \div 0.2 =$

$0.06 \div 20 =$

$0.6 \div 0.2 =$

$600 \div 0.2 =$

$0.06 \times 0.002 =$

$0.6 \times 2000 =$

$60 \times 0.2 =$

$600\,000 \times 2\,000 =$

$6000 \div 200 =$

$60 \div 20 =$

$0.6 \div 20 =$

$6 \times 0.02 =$

$600 \div 200 =$

$600 \times 200 =$

$60 \div 0.002 =$

$6 \div 0.02 =$

$0.006 \times 20 =$

$0.6 \times 0.2 =$

$60 \times 20 =$

Number Investigation

Follow these instructions. Write down your results.
Can you explain what is going on?

- Start with 24.
  Work out $24 \div 0.5$ and 24×2
  What do you notice?

- Now try $24 \div 2$ and 24×0.5
  Write down what you notice.

- Will this always happen or does it work only for 24?
  (Try some other numbers.)

- 0.5 and 2 are a special pair of numbers.
  Can you find another pair of numbers like this?
  Can you find a rule for finding pairs of numbers that do this?

Number Investigation

Follow these instructions. Write down your results.
Can you explain what is going on?

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- Will this always happen or does it work only for 24?
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- 0.5 and 2 are a special pair of numbers.
  Can you find another pair of numbers like this?
  Can you find a rule for finding pairs of numbers that do this?

1.6 Fractions

- Fractions are very unusual in having different equivalent forms. This concept takes some getting used to. Could go around pupil by pupil saying a fraction that's equivalent to a chosen starting fraction, and make a long line of = signs and equivalent fractions across the board. Those who find it easier can choose more complicated equivalent fractions. Once you get into a pattern (e.g., $\frac{30}{40}=\frac{300}{400}=\frac{30000}{40000}$), ask the next person to break the pattern and do something different.
- Rectangular "cakes" are generally easier to divide up than circular ones, and you can make use of squared paper. Also, the same cake can be cut in perpendicular directions to show a fraction of a fraction.
- Although we can *multiply* or *divide* the top and bottom of a fraction by the same number to get an equivalent fraction, it doesn't work if you *add* or *subtract* an amount; e.g., $\frac{2}{3}\neq\frac{3}{4}$ by adding 1 to top and bottom. This is a difficult concept to grasp – see section 1.10 for help with this.
- Common denominators make fractions easy to compare, but so do common *numerators*. For example, if asked which is bigger out of $\frac{2}{7}$ and $\frac{4}{15}$, working out a common denominator of 7×15 is unnecessary. Making the 1st fraction into $\frac{4}{14}$ makes it easy to see that this is bigger, since $\frac{1}{14}$ is more than $\frac{1}{15}$, so four of them must be bigger than four of the other. (Another way to compare the size of two fractions is to convert them both into decimals: you can think of this as effectively making a common denominator of 1.)

1.6.1 Number-lines are a good way to begin, as they make the point that fractions are just numbers on a number-line (like decimals).

Avoid allowing early work on fractions to be dominated by "fractions of*". They're not "operators", they're just numbers.*

1.6.2 Puzzle pictures (colour in the answers to produce a picture).

Often popular and available in books.

1.6.3 **NEED** Squared or isometric paper.
Draw shapes illustrating particular fractions; e.g., make a star shape and colour in $\frac{5}{8}$ of it.

Can make these into posters and use for display.

Pupils can make the shapes and the fractions as complicated as they like.

Differentiation by outcome.

1.6.4 Letters in words. What fraction of the word "letter" is "r", "t", "l", "e"? – total must be 1. (Pupils can use their names for these.)

Answers: $\frac{1}{6}$, $\frac{1}{3}$, $\frac{1}{6}$ and $\frac{1}{3}$. Easy to see that $\frac{2}{6}=\frac{1}{3}$ by grouping the letters in pairs. So it's useful to begin with words where repeated letters are adjacent "double-letters".

You can do a lot with this; e.g., "Find a word in which 'e' is $\frac{1}{2}$." Can make it easier by saying how many letters have to be in the word.

Answer: there are obviously many possible answers to these; e.g., "be", "been", etc.

This has a natural link to probability.

1.6.5 False Cancelling.
You can "cancel the 9's" in $\frac{19}{95}$ to get $\frac{1}{5}$, and these two fractions are the same size.
Find some others which work.

Answers:

$$\frac{1\cancel{6}}{\cancel{6}4}=\frac{1}{4},\quad \frac{2\cancel{6}}{\cancel{6}5}=\frac{2}{5},\quad \frac{4\cancel{9}}{\cancel{9}8}=\frac{4}{8}=\frac{1}{2}$$

1.6.6 **NEED** sets of cards containing different fractions (see pages – 1st page is easier; 2nd page is harder). In pairs, pupils shuffle the cards and place them face down on the table in a 4 × 4 grid.
Taking turns, each pupil turns over a pair of cards. If they are equivalent fractions he/she keeps them, otherwise they are turned back and left in the same positions. The winner is the one to collect the most pairs of cards.

If possible, photocopy each duplicate set onto a different coloured piece of card, so that different sets don't get muddled up, or write a different letter of the alphabet on the back of the cards in each different set, so pupils can check they have 16 "A" cards (or whatever) before they start.

1.6.7 Find out how long a "number 3" haircut is.

Answer: each "number" corresponds to $\frac{1}{8}$ inch, so a "number 3" is supposed to be $\frac{3}{8}$ inch long.

1.6.8 There is a pile of sweets on the table. I come into the room and eat one sweet. I take a third of what's left and put them in my pocket for later.
The second person comes into the room and looks at the pile of sweets that's left. She eats 1, takes a third of the rest and puts them in her pocket and leaves the room. Finally, the third person comes into the room and does the same.
Afterwards, we divide the sweets that are left on the table equally among all three of us.
How many sweets were there to begin with?
How many do we each end up with?

(At every point we are always talking about a whole number of sweets.)

Answer: more than one possible answer; e.g., 25 sweets

$25 \rightarrow 24 \rightarrow 16 \rightarrow 15 \rightarrow 10 \rightarrow 9 \rightarrow 6 \rightarrow 2$ each.

Each person has eaten 1. The 1st person has kept 8 + 2 = 10; the 2nd person has kept 5 + 2 = 7; the 3rd person has kept 3 + 2 = 5; and 3 + 10 + 7 + 5 = 25.

1.6.9 If I start with a fraction like $\frac{7}{38}$ and add 1 to the numerator and 1 to the denominator (so I end up with $\frac{8}{39}$), will the fraction get bigger or smaller?
Will this always happen?
Can you convince us why?

What if I add a number k to the top and bottom? What if k is negative/decimal/fractional?

Answer: bigger
Many different ways of arguing this one. Could say that adding 1 to the 7 makes a proportionally bigger increase than does adding 1 to 38, so the numerator is increasing by a bigger proportion than the denominator. This will always happen with a "bottom-heavy" fraction like this.
With a "top-heavy" fraction, this process will make the fraction smaller for a similar reason.
The same argument will apply when any positive number is added.
Algebraically, a common denominator between $\frac{a}{b}$ and $\frac{a+k}{b+k}$ is $b(b+k)$, so $\frac{a}{b}=\frac{a(b+k)}{b(b+k)}$ and $\frac{a+k}{b+k}=\frac{b(a+k)}{b(b+k)}$ and $b(a+k)>a(b+k)$ because $b>a$ for a "bottom-heavy" fraction (assuming $k>0$).

For $a=b$, the two fractions are equivalent; e.g., $\frac{3}{3}=\frac{4}{4}=\frac{11}{11}$.

If $k<0$ it must not equal $-b$, otherwise the denominator will be zero.

This can be useful when needing to compare, e.g., $\frac{13}{36}$ and $\frac{14}{37}$ to say which is bigger. The numerator has increased by a factor of $\frac{14}{13}$, whereas the denominator has increased by a factor of only $\frac{37}{36}$ (less), so $\frac{14}{37}$ must be bigger.

Easier set – photocopy onto coloured card (could laminate it) and guillotine.

$\frac{1}{2}$	$\frac{5}{10}$	$\frac{1}{5}$	$\frac{2}{10}$
$\frac{1}{3}$	$\frac{2}{6}$	$\frac{7}{10}$	$\frac{14}{20}$
$\frac{1}{4}$	$\frac{5}{20}$	$\frac{1}{9}$	$\frac{3}{27}$
$\frac{3}{10}$	$\frac{30}{100}$	$\frac{2}{5}$	$\frac{6}{15}$

Harder set – photocopy onto coloured card (could laminate it) and guillotine.

$\frac{2}{3}$	$\frac{4}{6}$	$\frac{3}{4}$	$\frac{9}{12}$
$\frac{7}{11}$	$\frac{14}{22}$	$\frac{5}{9}$	$\frac{15}{27}$
$\frac{1}{3}$	$\frac{20}{60}$	$\frac{2}{19}$	$\frac{4}{38}$
$\frac{3}{22}$	$\frac{115}{110}$	$\frac{2}{7}$	$\frac{10}{35}$

1.7 Fractions (Addition and Subtraction)

- Must be clear about the difference between the meaning of numerator numbers and denominator numbers. The numerator tells you how many there are. The denominator tells you what kind of thing you're talking about (like religious "denominations"). So in adding or subtracting fractions it doesn't make sense to add or subtract the denominators. (To do this is to treat fractions like column vectors, where both numbers in the vector represent "similar" things and, e.g., $\begin{pmatrix}1\\2\end{pmatrix}+\begin{pmatrix}2\\3\end{pmatrix}=\begin{pmatrix}3\\5\end{pmatrix}$.)

  $\frac{2}{7}+\frac{4}{7}=\frac{6}{7}$ is like $\frac{2}{cars}+\frac{4}{cars}=\frac{6}{cars}$ or $\frac{2}{sevenths}+\frac{4}{sevenths}=\frac{6}{sevenths}$ or $\frac{2}{£}+\frac{4}{£}=\frac{6}{£}$, meaning $£2+£4=£6$.

  It's worth spending time on this to make the point that the denominator is a kind of "label".
- Initially it's possible to allow *any* common denominator – having to find the LCM each time may over-complicate in the early stages.
- For addition, mixed numbers can be left as they are and the integers added separately. For subtraction, it's normally best to convert mixed numbers to "top-heavy" fractions, otherwise you have to deal in negative fractions in cases like $2\frac{1}{5}-1\frac{3}{4}$. To convert a mixed number, say $1\frac{2}{7}$, to a "top-heavy" fraction, think of it as $1+\frac{2}{7}=\frac{7}{7}+\frac{2}{7}=\frac{9}{7}$ and similarly (going the other way) to convert a "top-heavy" answer into a mixed number. Digits next to each other always mean addition (although adjacent *letters*, or number and letters, in algebra always mean multiplication); e.g., 34 means 30+4 and 4.8 means 4+0.8, so $1\frac{2}{7}$ means $1+\frac{2}{7}$.
- It's still worth using number-lines to reinforce that addition and subtraction are still just moving forwards and backwards along a normal number-line.

1.7.1 One way to begin by building on an awareness of equivalent fractions is to scatter some "random" fractions on the board and ask pupils in groups to pick pairs of fractions that they think will add up to *less than 1*. They should try to justify their choices to one another.

A collection of fractions might be the following (with their sums given in the grid). But write them onto the board in no particular order.

	$\frac{1}{5}$	$\frac{2}{9}$	$\frac{4}{11}$	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$
$\frac{1}{5}$	$\frac{2}{5}$							
$\frac{2}{9}$	$\frac{19}{45}$	$\frac{4}{9}$						
$\frac{4}{11}$	$\frac{31}{55}$	$\frac{58}{99}$	$\frac{8}{11}$					
$\frac{1}{2}$	$\frac{7}{10}$	$\frac{13}{18}$	$\frac{19}{22}$	1				
$\frac{3}{5}$	$\frac{4}{5}$	$\frac{37}{45}$	$\frac{53}{55}$	$1\frac{1}{10}$	$1\frac{1}{5}$			
$\frac{2}{3}$	$\frac{13}{15}$	$\frac{8}{9}$	$1\frac{1}{33}$	$1\frac{1}{6}$	$1\frac{4}{15}$	$1\frac{1}{3}$		
$\frac{3}{4}$	$\frac{19}{20}$	$\frac{35}{36}$	$1\frac{5}{44}$	$1\frac{1}{4}$	$1\frac{7}{20}$	$1\frac{5}{12}$	$1\frac{1}{2}$	
$\frac{4}{5}$	1	$1\frac{1}{45}$	$1\frac{9}{55}$	$1\frac{3}{10}$	$1\frac{2}{5}$	$1\frac{7}{15}$	$1\frac{11}{20}$	$1\frac{3}{5}$

Here there are sixteen pairs (including two the same) which add up to less than 1 (shaded above).

For example, pupils may say that $\frac{1}{2}+\frac{1}{5}<1$. because $\frac{1}{2}+\frac{1}{2}=1$ and $\frac{1}{5}<\frac{1}{2}$.

Another might be that $\frac{3}{4}+\frac{1}{5}<1$ because $\frac{3}{4}+\frac{1}{4}=1$ and $\frac{1}{5}<\frac{1}{4}$.

More sophisticated arguments are also possible.

There may be certain pairs left over that it isn't possible to decide about – so we need a way of working out the total of two fractions.

Can you find more than two fractions whose total is less than 1?

Many possibilities.

1.7.2 Puzzle pictures (colour in the answers to produce a picture).

Often popular and available in books.

1.7.3 It is possible to use the digits from 1 to 9 once each to make fractions that add up to 1.
How do you do it?
$\frac{?}{12}+\frac{?}{34}+\frac{?}{68}=1$

Answer: $\frac{9}{12}+\frac{5}{34}+\frac{7}{68}=1$

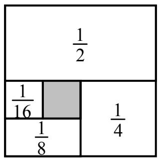
Also, using zero as well, $\frac{35}{70}+\frac{148}{296}=1$ and $96+\frac{2148}{537}=100$ and others.

1.7.4 Magic squares with fractions (see sheet).

Photocopy the sheet or copy some onto the board.

1.7.5 Patterns in adding fractions.
Work out $\frac{1}{2}+\frac{1}{4}$, then $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$, then $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$, and so on.
What is happening?

Answer:
The total ($\frac{3}{4}$*,* $\frac{7}{8}$*,* $\frac{15}{16}$*) gets closer and closer to 1.*
It's easy to see from the drawing below that the total area is approaching the 1 × 1 square.



Try a different pattern, still dividing by a constant amount each time;
e.g., $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots$
What does the total get closer and closer to?
Can you summarise what will happen?
What if you started with $\frac{1}{10}$?

In general for a geometric series $\sum_{n=1}^{\infty}\left(\frac{1}{x}\right)^n=\frac{1}{x-1}$ *for* $|x|>1$*, so if* $x=3$*, then the sum approaches* $\frac{1}{2}$ *as you add on more and more terms.*

And with $\frac{1}{10}$ the infinite sum is $\frac{1}{9}$.

1.7.6 Work these out.
What do you notice?

$$\frac{1+3}{5+7},\ \frac{1+3+5}{7+9+11},\ \frac{1+3+5+7}{9+11+13+15}$$

Answer: all equal $\frac{1}{3}$ *(the fraction that would fit at the beginning of the pattern).*

Galileo (1564-1642) noticed this.

1.7.7 Find the sums of interesting-looking sequences; e.g., $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}$ or even $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}$.

(The second one is too big for most calculators to handle!)

Answers: $1\frac{83}{140}$ *and* $2\frac{2341}{2520}$.
(There's no elegant short-cut like there is with adding the integers from 1 to 10 to get 55.)

After doing all the work involved in these, you're unlikely ever to think again that adding fractions is a simple matter of adding the numerators and adding the denominators!

Fractions Calculations

1. When *adding or subtracting* fractions, you always need to find a *common denominator*.
2. When *multiplying* fractions, you *don't* need to find a common denominator, but you should always *cancel* as much as possible before *multiplying the numerators (tops) and multiplying the denominators (bottoms)*.
3. When *dividing* fractions, leave the *first* fraction alone, turn the *second* fraction *upside down* and change the sign to *multiply*.
4. Change *mixed numbers* into "top-heavy" fractions for subtracting, multiplying and dividing.
5. Always *simplify* your final answer and turn "top-heavy" fractions into *mixed numbers*.

Magic Squares (Fractions)

Fill in the missing fractions in these *magic squares*.
In each square the total along every column, every row and both diagonals is the same.
Different squares have different totals. Write all the fractions in their simplest forms.

1

$\frac{4}{15}$		
$\frac{1}{5}$	$\frac{1}{3}$	
$\frac{8}{15}$		

2

$\frac{1}{10}$	$\frac{1}{4}$	$\frac{1}{25}$
	$\frac{13}{100}$	

3

$1\frac{2}{9}$		
	1	$\frac{5}{9}$
		$\frac{7}{9}$

4

5		
$6\frac{2}{3}$	$4\frac{1}{3}$	1

5

	$1\frac{4}{7}$	
	1	
	$\frac{3}{7}$	$1\frac{1}{7}$

6

		5
	$6\frac{1}{2}$	
8		11

Magic Squares (Fractions) **ANSWERS**

Fill in the missing fractions in these *magic squares*.
In each square the total along every column, every row and both diagonals is the same.
Different squares have different totals. Write all the fractions in their simplest forms.

1 magic total 1

$\frac{4}{15}$	$\frac{9}{15}$	$\frac{2}{15}$
$\frac{1}{5}$	$\frac{1}{3}$	$\frac{7}{15}$
$\frac{8}{15}$	$\frac{1}{15}$	$\frac{6}{15}$

2 magic total $\frac{39}{100}$

$\frac{1}{10}$	$\frac{1}{4}$	$\frac{1}{25}$
$\frac{7}{100}$	$\frac{13}{100}$	$\frac{19}{100}$
$\frac{11}{50}$	$\frac{1}{100}$	$\frac{4}{25}$

3 magic total 3

$1\frac{2}{9}$	$\frac{1}{9}$	$1\frac{6}{15}$
$1\frac{4}{15}$	1	$\frac{5}{9}$
$\frac{1}{3}$	$1\frac{8}{15}$	$\frac{7}{9}$

4 magic total 9

5	$1\frac{2}{3}$	$2\frac{1}{3}$
$\frac{1}{3}$	3	$5\frac{2}{3}$
$6\frac{2}{3}$	$4\frac{1}{3}$	1

5 magic total 3

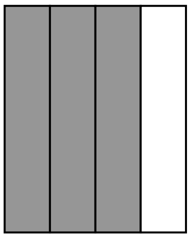
$\frac{6}{7}$	$1\frac{4}{7}$	$\frac{4}{7}$
$\frac{5}{7}$	1	$1\frac{2}{7}$
$1\frac{3}{7}$	$\frac{3}{7}$	$1\frac{1}{7}$

6 magic total $19\frac{1}{2}$

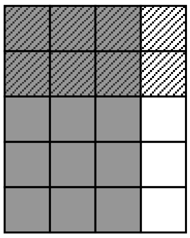
2	$12\frac{1}{2}$	5
$9\frac{1}{2}$	$6\frac{1}{2}$	$3\frac{1}{2}$
8	$\frac{1}{2}$	11

1.8 Fractions (Multiplication and Division)

- Multiplication of fractions is an easier *process* than addition and subtraction, although harder conceptually. We can just "define" multiplication of fractions as "multiply the tops, multiply the bottoms", but pupils need to see that this is a sensible definition in the context of integers. (The principle is that you can define anything you like in maths so long as it's useful and doesn't contradict anything you've already defined.)
- Area provides a useful context: a room $2\frac{1}{2}$ m by 3 m will clearly have an area $1\frac{1}{2}$ m^2 bigger than a room only 2 m by 3 m. (A drawing makes this clear.)
- To see that $\frac{2}{5}\times\frac{3}{4}=\frac{3}{10}$, draw a rectangle and divide it into 4 equal *vertical* strips. Shade 3.



This represents $\frac{3}{4}$ of the total area.
Then divide the whole rectangle into 5 equal *horizontal* strips. Shade 2 of them.



Where the shadings overlap there are 6 twentieths of the whole rectangle; and you can see that this is the same as $\frac{3}{10}$. So $\frac{2}{5}$ of $\frac{3}{4}$ is $\frac{6}{20}$, or $\frac{3}{10}$.

- It's important to make a habit of cancelling as much as possible before multiplying so that you have easier multiplications to do and less simplifying at the end (none if you cancel as much as possible).
- For mixed problems involving + – × ÷, BIDMAS becomes essential.

1.8.1 A possible context for multiplication would be winning £$\frac{3}{4}$ m in a competition and wanting to share it equally between yourself and a friend.

Answer: $\frac{1}{2}$ *of* £$\frac{3}{4}$ m is £$\frac{3}{8}$ m = £375 k
"$\frac{1}{2}$ *of" is the same as "*$\div 2$*"*

1.8.2 A possible context for division by a fraction could be making a rice pudding. It takes $\frac{3}{4}$ of a pint of milk to make a rice pudding. How many rice puddings can I make with 18 pints of milk?

Pupils could discuss in groups what they think the answer is and why.

Pupils can check afterwards that calculators agree.

Answer: It should be clear that the question is how many $\frac{3}{4}$ *'s go into 18, so we need to work out* $18\div\frac{3}{4}$*, and the answer is 24.*
One approach is to see that $18\div\frac{1}{4}=18\times 4$ *(how many quarter pints can you get out of 18 pints) and that* $18\div\frac{3}{4}$ *will be one third of this amount.*
Another approach is to view division as repeated subtraction and see how many times you can take away $\frac{3}{4}$ *from 18 before all the milk has gone.*

1.8.3 Puzzle pictures (colour in the answers to produce a picture).

Often popular and available in books.

1.8.4 Sometimes you can divide fractions without needing the "turn upside down and multiply" rule; e.g., it can be seen that $\frac{3}{5} \div \frac{4}{5} = \frac{3}{4}$, because "3 anythings, divided by 4 of the same must make $\frac{3}{4}$".
Some *divisions* can be done fairly easily in this way by finding *common denominators*;
e.g., $\frac{2}{3} \div \frac{4}{5} = \frac{10}{15} \div \frac{12}{15} = \frac{10}{12} = \frac{5}{6}$.

This is a good opportunity to see that there is frequently more than one way of solving a problem, and that different methods are better in different circumstances.

This may be longer but makes more sense to some pupils, although it would lead to unnecessary work if you had, say, $\frac{1}{19} \div \frac{1}{20}$.

1.8.5 It's helpful to see some patterns in multiplying and dividing different fractions (see "Multiplying and Dividing Fractions" sheet).

Explaining why the patterns (e.g., $\frac{1}{3}$ times table) come where they do is challenging.

This helps to see that the answers fit in with the answers to multiplying and dividing integers. This reinforces the idea that fractions are just numbers.

Not every one has to be worked out once the patterns are spotted.

1.8.6 Work out $(1+\frac{1}{2})(1+\frac{1}{3})(1+\frac{1}{4})(1+\frac{1}{5})$.

Answer:
$$\frac{\cancel{3}}{2} \times \frac{\cancel{4}}{\cancel{3}} \times \frac{\cancel{5}}{\cancel{4}} \times \frac{6}{\cancel{5}} = \frac{6}{2} = 3,$$
cancelling before multiplying.

1.8.7 A spider climbs up out of a bath 50 cm tall. Each day he climbs up $\frac{2}{3}$ cm, and each night he slips back $\frac{1}{4}$ cm. If he starts at 6 am (the beginning of the day) on the first day, on which day will he make it completely out of the bath?

If the spider starts at the beginning of day 1 (say 6 am) and climbs at a steady rate during the days (still slipping back at night) at what time on the final day will he make it over the side of the bath?

Answer: day 120. After each day and night he ends up $\frac{2}{3} - \frac{1}{4} = \frac{5}{12}$ cm further up the bath, so $50 \div \frac{5}{12} = 120$, but this includes slipping back at the end of the 120th day.
So to be more accurate, at the end of the 119th day he will have achieved $119 \times \frac{5}{12} = 49\frac{7}{12}$ cm, so he has only $\frac{5}{12}$ cm to go. This will take $\frac{5}{12} \div \frac{2}{3}$ of a day $= \frac{5}{8}$ of a day $= \frac{5}{8} \times 12$ hours through the day = $7\frac{1}{2}$ hours after 6 am = 1.30 pm on the 120th day.

1.8.8 Pupils can "invent" rules for multiplying and dividing fractions by trying to make their behaviour fit with what they know happens with decimals.
e.g., $0.5 \times 0.5 = 0.25$ according to knowledge / calculator / "boxes method" / whatever.
What rule can we use with $\frac{1}{2} \times \frac{1}{2}$ to make it give us $\frac{1}{4}$?

A calculator that will not do fractions may be useful.
(Decimals are "decimal fractions" anyway.)

It doesn't matter if pupils don't recognise what fraction is equivalent to the decimal answers they obtain: they can experiment with different fractions to see which turns out to be equivalent (numerator ÷ denominator on the calculator).

1.8.9 There is a natural link with probability work: When should you multiply probabilities and when can't you?

Probabilities are usually written as fractions or decimals.

You always *multiply probabilities along the branches of a tree diagram, because the probability written on any branch is always* conditional *on the necessary events happening along the branches that lead to that branch.*

(See section 3.5 for more on this.)

Answer:
When you want the probability of two events A and B both *happening, you multiply:*
$p(A \cap B) = p(A)p(B|A) = p(B)p(A|B)$, *where $p(A|B)$ means the conditional probability that A happens given that B has already happened.*
If A and B are independent events – i.e., the chance of A happening is the same whether or not B happens – then these formulas reduce to $p(A \cap B) = p(A)p(B)$, because $p(B|A) = p(B|\overline{A}) = p(B)$, etc.

Multiplying and Dividing Fractions

Fill in the missing numbers in these grids. Write all the answers as mixed numbers with fractions in their lowest forms.

(horizontal number *multiplied* by vertical number)

$\times$	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5
$\frac{1}{2}$	$\frac{1}{4}$									
1			$1\frac{1}{2}$							
$1\frac{1}{2}$										
2										
$2\frac{1}{2}$										
3										
$3\frac{1}{2}$										
4										
$4\frac{1}{2}$										
5										

(*horizontal* number *divided* by *vertical* number each time)

$\div$	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5
$\frac{1}{2}$	1	2								
1	$\frac{1}{2}$									
$1\frac{1}{2}$	$\frac{1}{3}$									
2										
$2\frac{1}{2}$										
3										
$3\frac{1}{2}$										
4										
$4\frac{1}{2}$										
5										

Extra Task Make up a number square like this with the fractions around the edge going up in $\frac{1}{3}$'s or $\frac{1}{4}$'s or some other fraction.

Multiplying and Dividing Fractions **ANSWERS**

Fill in the missing numbers in these grids. Write all the answers as mixed numbers with fractions in their lowest forms.

(horizontal number *multiplied* by vertical number)

×	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$1\frac{1}{4}$	$1\frac{1}{2}$	$1\frac{3}{4}$	2	$2\frac{1}{4}$	$2\frac{1}{2}$
1	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5
$1\frac{1}{2}$	$\frac{3}{4}$	$1\frac{1}{2}$	$2\frac{1}{4}$	3	$3\frac{3}{4}$	$4\frac{1}{2}$	$5\frac{1}{4}$	6	$6\frac{3}{4}$	$7\frac{1}{2}$
2	1	2	3	4	5	6	7	8	9	10
$2\frac{1}{2}$	$1\frac{1}{4}$	$2\frac{1}{2}$	$3\frac{3}{4}$	5	$6\frac{1}{4}$	$7\frac{1}{2}$	$8\frac{3}{4}$	10	$11\frac{1}{4}$	$12\frac{1}{2}$
3	$1\frac{1}{2}$	3	$4\frac{1}{2}$	6	$7\frac{1}{2}$	9	$10\frac{1}{2}$	12	$13\frac{1}{2}$	15
$3\frac{1}{2}$	$1\frac{3}{4}$	$3\frac{1}{2}$	$5\frac{1}{4}$	7	$8\frac{3}{4}$	$10\frac{1}{2}$	$12\frac{1}{4}$	14	$15\frac{3}{4}$	$17\frac{1}{2}$
4	2	4	6	8	10	12	14	16	18	20
$4\frac{1}{2}$	$2\frac{1}{4}$	$4\frac{1}{2}$	$6\frac{3}{4}$	9	$11\frac{1}{4}$	$13\frac{1}{2}$	$15\frac{3}{4}$	18	$20\frac{1}{4}$	$22\frac{1}{2}$
5	$2\frac{1}{2}$	5	$7\frac{1}{2}$	10	$12\frac{1}{2}$	15	$17\frac{1}{2}$	20	$22\frac{1}{2}$	25

(*horizontal* number *divided* by *vertical* number each time)

÷	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5
$\frac{1}{2}$	1	2	3	4	5	6	7	8	9	10
1	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5
$1\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{3}$	1	$1\frac{1}{3}$	$1\frac{2}{3}$	2	$2\frac{1}{3}$	$2\frac{2}{3}$	3	$3\frac{1}{3}$
2	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$1\frac{1}{4}$	$1\frac{1}{2}$	$1\frac{3}{4}$	2	$2\frac{1}{4}$	$2\frac{1}{2}$
$2\frac{1}{2}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	1	$1\frac{1}{5}$	$1\frac{2}{5}$	$1\frac{3}{5}$	$1\frac{4}{5}$	2
3	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	1	$1\frac{1}{6}$	$1\frac{1}{3}$	$1\frac{1}{2}$	$1\frac{2}{3}$
$3\frac{1}{2}$	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{3}{7}$	$\frac{4}{7}$	$\frac{5}{7}$	$\frac{6}{7}$	1	$1\frac{1}{7}$	$1\frac{2}{7}$	$1\frac{3}{7}$
4	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{7}{8}$	1	$1\frac{1}{8}$	$1\frac{1}{4}$
$4\frac{1}{2}$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{1}{3}$	$\frac{4}{9}$	$\frac{5}{9}$	$\frac{2}{3}$	$\frac{7}{9}$	$\frac{8}{9}$	1	$1\frac{1}{9}$
5	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{3}{10}$	$\frac{2}{5}$	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{7}{10}$	$\frac{4}{5}$	$\frac{9}{10}$	1

Extra Task Make up a number square like this with the fractions around the edge going up in $\frac{1}{3}$'s or $\frac{1}{4}$'s or some other fraction.

1.9 Percentages

- There's nothing fundamental about dividing by 100 – we choose to invent % because it's convenient. Pupils need to see this convenience early on (e.g., task 1.9.1). For very small proportions it's more convenient to use ppm (parts per million) instead.
- % increase and decrease seem to work best using "decimal multipliers"; i.e., to increase by 12% multiply by 1.12 and to decrease by 12% multiply (not divide) by 0.88.
- To work out the % increase or decrease given the starting amount and finishing amount (or either and the difference), always work out $\frac{\text{new amount}}{\text{old amount}}$ (this is the "multiplier"). If this comes to, say, 1.32, it's been a 32% increase. To find an "old amount" from the "new amount" (the trickiest situation) you need to *divide by the multiplier.* (The sheet "Percentage Increase and Decrease" assumes this approach.)

1.9.1 Write on the board the marks I might have got at school in my report:

Eng $\frac{6}{10}$	Maths $\frac{19}{25}$	Sci $\frac{19}{30}$
History $\frac{9}{20}$	Geog $\frac{9}{15}$	RE $\frac{35}{50}$
PE $\frac{2}{3}$	Music $\frac{4}{8}$	French $\frac{3}{5}$

Converting them all into %'s makes it much easier to make comparisons.
Calculators not really needed.

Eng 60%	*Maths 76%*	*Sci 63%*
History 45%	*Geog 60%*	*RE 75%*
PE 67%	*Music 50%*	*French 60%*

Imagine I go home and my parents say, "You're obviously good at Geography, because you got 9, but not so good at PE because you only got 2".
Why might that not be right?

Of course, the tests in the different subjects might not be of the same "difficulty". (How can you compare "difficulty" in two different subjects, anyway? We might want to know what the class median was in each, though)

What's my best/worst subject?
What am I better at – history or geog? maths or science?
Put my subjects in order.

Order (best to worst): Maths-76 (of course!), RE-75, PE-67, Sci-63, Eng=Geog=French-60, Music-50, History-45.

It's helpful to discuss comparisons; e.g., $\frac{9}{15}$ must be better than $\frac{9}{20}$, etc.

1.9.2 Discuss where you come across percentages apart from in maths lessons; e.g., how are they used in shops?

10% off price, "100% fat free", "100% cotton", etc., exams, pie charts.

What do you think about a disinfectant that says "Kills 101% of germs"?

This is just hyperbole – not mathematically accurate.

What about a sports coach who says he wants the players to "give 110%"?

110% may not be as silly as it sounds – it may mean give 10% more than usual (10% increase).

Music exams are out of 150 but people sometimes say "percent" if the mark is under 100, yet it isn't really a %.

Can % be more than 100 or is that impossible? When does it make sense to have more than 100% and when does it not make sense?

It is possible when something ends up bigger than it started; e.g., "the crowd this year is 120% of what it was last year" means a 20% increase.
Drawing a number-line from 0 to about 350% and marking decimals 0 to 3.5 is helpful in seeing that %'s are just numbers.

What's the difference between "increased *by* 200% of its original value" and "increased *to* 200% of its original value"?

"by" means now 3 × what it was
"to" means now 2 × what it was

1.9.3	If the base of a rectangle is increased by 10% and the area is unchanged, by what percentage is the width *decreased*?	*Answer:* $\frac{1}{1.1} = 0.909...$*, so a decrease of about 0.09 or 9%.*
1.9.4	The difference between increasing a number by 7% and decreasing it by 8% is 30. What is the number? Pupils can make up some puzzles like this.	*Answer: 15% of the number = 30, so number = 30 ÷ 0.15 =* ***200*** *(check: 214 – 184 = 30)*
1.9.5	The original price of an item is £15. After two mark-ups the final price is £20.70. If the 1st mark-up is 15%, what is the other? What if it was the 2nd mark-up that was 15%? What would the 1st have been?	*Easy with multipliers:* *15.00 × 1.15 × m = 20.70* *m = 1.2, so the mark-up is 20%.* *It doesn't matter – you get the same answer.*
1.9.6	Which is bigger 72% of £48 or 48% of £72? *This can be useful because a hard % of an easy number can be written as an easy % of a hard number; e.g., 73% of £50 = 50% of £73 = £36.50.*	*They're the same, but can pupils explain why? One way to look at it is that* *0.72 × 48 = 0.48 × 72*
1.9.7	If I decrease an amount by 10% and then increase the new amount by 10% have I got back the amount I started with? (Easy to embellish with a context!) What if I do it in the opposite order?	*No – the 1st 10% was 10% of a bigger amount than the 2nd 10% was, so what I added on was smaller than what I took away.* *If I started with* p *I now have 1.1 × 0.9 ×* p *= 0.99 ×* p *(i.e., I've lost 1% of what I had).* *This time it's 0.9 × 1.1 ×* p *= 0.99 ×* p*; i.e., the same answer.*
1.9.8	If 60% of the school like swimming, 65% like football, 70% like rugby, 75% like hockey and 80% like basketball, what percentage like them all? *Could draw a Venn (1834-1923) diagram.*	*Answer: Total = 60+65+70+75+80 = 350%. If you share them out as equally as possible everyone has to like at least 3, and at least 50% have to like all four. The maximum % that could like all four would be 60%, so the answer is between 50% and 60%.*
1.9.9	Carbon Monoxide (CO) is a poisonous gas which is found in the air and which some people are concerned about. Carbon monoxide detectors measure the concentration of CO in the air in ppm (parts per million). What is 100 ppm as a percentage?	*Answer: 1 ppm is* $\frac{1}{1000000}$ *= 0.0001%, so 10 000 ppm = 1% and 100 ppm = 0.01%.* *A more poisonous gas is hydrogen cyanide (HCN) which is dangerous at a level of 10 ppm = 0.001%.*
1.9.10	VAT. A ready reckoner (see sheet) can be useful in a situation where you always want the same percentage of different amounts.	*Conveniently, 17.5% = 10% + 5% + 2.5%, and each of these is half the previous amount. This can help with non-calculator calculations.*
1.9.11	Work out the percentage of your life you have spent on different things. e.g., percentage of your life asleep/watching TV/at parties/shopping/doing a particular sport/practising a musical instrument, etc.; percentage of the school year which is holiday; percentage of school lesson time given over to maths (if year 7, compare with primary school); percentage of your life spent in assemblies; percentage of your life you've spent as a teenager; percentage of your life spent in a particular country (if relevant).	*Some of these aren't too hard, because you can work out per 24 hours or per week.* *It's worth thinking through the kinds of assumptions we make when doing these sorts of estimates.*

Percentage Increase and Decrease

Fill in the gaps in the table.
The first one is done already.

	old price	new price	$\frac{\text{new price}}{\text{old price}}$	what's happened?
1	£34.00	£50.00	1.47	47% increase
2	£6.50	£7.20		
3	£8.50	£8.10		
4	£241.00			41% decrease
5	£78.20			4% increase
6	£1.60		1.24	
7	£852.10		0.30	
8	£29.00			32% decrease
9	£43.80			90% increase
10	£329.35	£400.00		
11	£22.00	£10.00		
12		£179.00	0.90	10% decrease
13		£4.00	1.15	15% increase
14		£11.00		16% increase
15		£11.11		8% decrease
16		£2.00		33% decrease
17		£1,499.00		17% increase
18		£8.50	1.09	
19	£8.00			10% decrease
20		£543.00		17% decrease

Percentage Increase and Decrease **ANSWERS**

Fill in the gaps in the table.
The first one is done already.

	old price	new price	$\frac{\text{new price}}{\text{old price}}$	what's happened?
1	£34.00	£50.00	1.47	47% increase
2	£6.50	£7.20	**1.11**	**11% increase**
3	£8.50	£8.10	**0.95**	**5% decrease**
4	£241.00	**£142.19**	**0.59**	41% decrease
5	£78.20	**£81.33**	**1.04**	4% increase
6	£1.60	**£1.98**	1.24	**24% increase**
7	£852.10	**£255.63**	0.30	**70% decrease**
8	£29.00	**£19.72**	**0.68**	32% decrease
9	£43.80	**£83.22**	**1.90**	90% increase
10	£329.35	£400.00	**1.21**	**21% increase**
11	£22.00	£10.00	**0.45**	**55% decrease**
12	**£198.89**	£179.00	0.90	10% decrease
13	**£3.48**	£4.00	1.15	15% increase
14	**£9.48**	£11.00	**1.16**	16% increase
15	**£12.08**	£11.11	**0.92**	8% decrease
16	**£2.99**	£2.00	**0.67**	33% decrease
17	**£1,281.20**	£1,499.00	**1.17**	17% increase
18	**£7.80**	£8.50	1.09	**9% increase**
19	£8.00	**£7.20**	**0.90**	10% decrease
20	**£654.22**	£543.00	**0.83**	17% decrease

Ready Reckoner for 17.5%

17.5% **17.5%**

£	£	£	£	£	£	£	£	£	£	p	p	p	p
1	0.18	**51**	8.93	**101**	17.68	**152**	26.60	**310**	54.25	**1**	0	**51**	9
2	0.35	**52**	9.10	**102**	17.85	**154**	26.95	**320**	56.00	**2**	0	**52**	9
3	0.53	**53**	9.28	**103**	18.03	**156**	27.30	**330**	57.75	**3**	1	**53**	9
4	0.70	**54**	9.45	**104**	18.20	**158**	27.65	**340**	59.50	**4**	1	**54**	9
5	0.88	**55**	9.63	**105**	18.38	**160**	28.00	**350**	61.25	**5**	1	**55**	10
6	1.05	**56**	9.80	**106**	18.55	**162**	28.35	**360**	63.00	**6**	1	**56**	10
7	1.23	**57**	9.98	**107**	18.73	**164**	28.70	**365**	63.88	**7**	1	**57**	10
8	1.40	**58**	10.15	**108**	18.90	**166**	29.05	**370**	64.75	**8**	1	**58**	10
9	1.58	**59**	10.33	**109**	19.08	**168**	29.40	**380**	66.50	**9**	2	**59**	10
10	1.75	**60**	10.50	**110**	19.25	**170**	29.75	**390**	68.25	**10**	2	**60**	11
11	1.93	**61**	10.68	**111**	19.43	**172**	30.10	**400**	70.00	**11**	2	**61**	11
12	2.10	**62**	10.85	**112**	19.60	**174**	30.45	**410**	71.75	**12**	2	**62**	11
13	2.28	**63**	11.03	**113**	19.78	**175**	30.63	**420**	73.50	**13**	2	**63**	11
14	2.45	**64**	11.20	**114**	19.95	**176**	30.80	**430**	75.25	**14**	2	**64**	11
15	2.63	**65**	11.38	**115**	20.13	**178**	31.15	**440**	77.00	**15**	3	**65**	11
16	2.80	**66**	11.55	**116**	20.30	**180**	31.50	**450**	78.75	**16**	3	**66**	12
17	2.98	**67**	11.73	**117**	20.48	**182**	31.85	**460**	80.50	**17**	3	**67**	12
18	3.15	**68**	11.90	**118**	20.65	**184**	32.20	**470**	82.25	**18**	3	**68**	12
19	3.33	**69**	12.08	**119**	20.83	**186**	32.55	**480**	84.00	**19**	3	**69**	12
20	3.50	**70**	12.25	**120**	21.00	**188**	32.90	**490**	85.75	**20**	4	**70**	12
21	3.68	**71**	12.43	**121**	21.18	**190**	33.25	**500**	87.50	**21**	4	**71**	12
22	3.85	**72**	12.60	**122**	21.35	**192**	33.60	**550**	96.25	**22**	4	**72**	13
23	4.03	**73**	12.78	**123**	21.53	**194**	33.95	**600**	105.00	**23**	4	**73**	13
24	4.20	**74**	12.95	**124**	21.70	**196**	34.30	**650**	113.75	**24**	4	**74**	13
25	4.38	**75**	13.13	**125**	21.88	**198**	34.65	**700**	122.50	**25**	4	**75**	13
26	4.55	**76**	13.30	**126**	22.05	**200**	35.00	**750**	131.25	**26**	5	**76**	13
27	4.73	**77**	13.48	**127**	22.23	**204**	35.70	**800**	140.00	**27**	5	**77**	13
28	4.90	**78**	13.65	**128**	22.40	**205**	35.88	**850**	148.75	**28**	5	**78**	14
29	5.08	**79**	13.83	**129**	22.58	**210**	36.75	**900**	157.50	**29**	5	**79**	14
30	5.25	**80**	14.00	**130**	22.75	**215**	37.63	**950**	166.25	**30**	5	**80**	14
31	5.43	**81**	14.18	**131**	22.93	**216**	37.80	**1000**	175.00	**31**	5	**81**	14
32	5.60	**82**	14.35	**132**	23.10	**220**	38.50	**1016**	177.80	**32**	6	**82**	14
33	5.78	**83**	14.53	**133**	23.28	**225**	39.38	**1250**	218.75	**33**	6	**83**	15
34	5.95	**84**	14.70	**134**	23.45	**228**	39.90	**1500**	262.50	**34**	6	**84**	15
35	6.13	**85**	14.88	**135**	23.63	**230**	40.25	**1728**	302.40	**35**	6	**85**	15
36	6.30	**86**	15.05	**136**	23.80	**235**	41.13	**1750**	306.25	**36**	6	**86**	15
37	6.48	**87**	15.23	**137**	23.98	**240**	42.00	**1760**	308.00	**37**	6	**87**	15
38	6.65	**88**	15.40	**138**	24.15	**245**	42.88	**2000**	350.00	**38**	7	**88**	15
39	6.83	**89**	15.58	**139**	24.33	**250**	43.75	**2240**	392.00	**39**	7	**89**	16
40	7.00	**90**	15.75	**140**	24.50	**255**	44.63	**2500**	437.50	**40**	7	**90**	16
41	7.18	**91**	15.93	**141**	24.68	**260**	45.50	**3000**	525.00	**41**	7	**91**	16
42	7.35	**92**	16.10	**142**	24.85	**265**	46.38	**4000**	700.00	**42**	7	**92**	16
43	7.53	**93**	16.28	**143**	25.03	**270**	47.25	**5000**	875.00	**43**	8	**93**	16
44	7.70	**94**	16.45	**144**	25.20	**275**	48.13	**6000**	1050.00	**44**	8	**94**	16
45	7.88	**95**	16.63	**145**	25.38	**280**	49.00	**7000**	1225.00	**45**	8	**95**	17
46	8.05	**96**	16.80	**146**	25.55	**285**	49.88	**8000**	1400.00	**46**	8	**96**	17
47	8.23	**97**	16.98	**147**	25.73	**288**	50.40	**9000**	1575.00	**47**	8	**97**	17
48	8.40	**98**	17.15	**148**	25.90	**290**	50.75	**10000**	1750.00	**48**	8	**98**	17
49	8.58	**99**	17.33	**149**	26.08	**295**	51.63	**15000**	2625.00	**49**	9	**99**	17
50	8.75	**100**	17.50	**150**	26.25	**300**	52.50	**20000**	3500.00	**50**	9	**100**	18

1.10 Ratio

- Ratio can be a comparison between 2 quantities or more than 2 quantities. Recipes are a possible starting point; e.g., pancakes (4 tbsp plain flour, pinch of salt, 1 egg, 300 ml milk). How much mixture could you make with 1 pint (about 600 ml) milk? How many eggs would you need, etc.? An example like this can destroy "linear thinking": "500g flour and 1 egg" – if I wanted to make a larger amount would I really use "501g flour and 2 eggs"? or "600g flour and 101 eggs"?! Why doesn't this work?
- Fizzy orange: You can buy orangeade in the shops but it tastes much better if you make it yourself out of real orange juice from a carton and lemonade. Imagine we're making some for this class. How much do you think we would need? 6 litres? Suppose I use 2 litres of orange with 4 litres of lemonade. Imagine that isn't going to be enough, and I want 8 litres, so instead I use 3 litres of orange with 5 litres of lemonade? (Write up these possibilities on the board – see table below.) Will it taste the same?
  (Some pupils may argue that it will because you've added the same extra amount of orange as lemonade.) If not, will it taste more 'orangey' or more fizzy? Be awkward – "Are you saying it will taste more orangey because it's got more orange in it? But it's got more lemonade in it as well."
  Answer: 2nd one more orangey because $\frac{3}{8} > \frac{1}{3}$. *Expect arguments like this: "You've added the same amount of lemonade as orange but it has less effect because there was more lemonade to start with. Imagine 1 litre of orange with 99 litres of lemonade. 1 more litre of orange will make it nearly twice as orangey but one more litre of lemonade won't make any noticeable difference."*
  If pupils start talking about 3 litres of orange and 6 litres of lemonade, or other possibilities, write them up in a table like this:

	A	***B***	***C***
orange	*2*	*3*	*3*
lemonade	*4*	*5*	*6*
total	*6*	*8*	*9*

  When you've got A, B, C, D, E, etc., decide which will taste the same. Could make a scale of "orangeyness" going one way and fizziness going the other way and decide where to put A, B, C, …
  This can be a very useful context because it's clear what is the same (the taste) when the "ratio" is the same. This is also true with the next example (colour).
- Paint: I've made light blue paint by mixing 1 tin of white with 2 tins of blue. I need 1 more tin of light blue. How much blue and white paint should I use to make it exactly the same shade of blue?
  Answer: $\frac{1}{3}$ *tin of white with* $\frac{2}{3}$ *tin of blue.*
- Beads on a necklace – red and green: RRGGGRRGGGRRGGG. Ratio can involve discrete items as well as continuous amounts.

1.10.1 Scaling up recipes.
Cake: 4 oz (100 g) flour; 4 oz (100 g) sugar; 4 oz (100 g) margarine; 2 eggs.
Pastry: 8 oz (225 g) flour; 2 oz (50 g) lard; 2 oz (50 g) margarine; 2 tbsp water.
Biscuits: 8 oz (225 g) flour; 4 oz (100 g) sugar; 4 oz (100 g) margarine; 1 egg.
Shortbread: 9 oz (250 g) flour; 3 oz (75 g) sugar; 6 oz (175 g) butter.

Imagine making enough for the whole class. Generally that will be an awkward number, which raises the issue of how accurate we need to be ("5.6667 pinches of salt", etc.!).

What about the oven temperature and the cooking time?

They may need to be higher/longer for a larger recipe, but you certainly don't want to scale them up in the same proportion as the ingredients! (Scaling up temperature is a nonsense anyway, as it makes a difference whether you use °C or °F!) See if pupils can explain why it's silly. Could imagine two identical ovens next to each other in the kitchen cooking identical cakes. Then combine into one big cake in one big oven.

In fact for a larger quantity you might go for a lower temperature and longer cooking time.

1.10.2 Puzzle pictures (colour in the answers to produce a picture).

Often popular and available in books.

1.10.3 It takes $1\frac{1}{2}$ hens $1\frac{1}{2}$ days to lay $1\frac{1}{2}$ eggs. How long does it take 2 hens to lay 4 eggs?

Many problems can be solved in a similar way, using a table like this.

Answer: 3 days (since 2 hens between them lay 2 eggs every $1\frac{1}{2}$ days).

Can use a table to make it clear, always keeping 1 thing the same as you go to the next line.

hens	***days***	***eggs***
$1\frac{1}{2}$	$1\frac{1}{2}$	$1\frac{1}{2}$
1	$1\frac{1}{2}$	1
2	$1\frac{1}{2}$	2
2	**3**	4

1.10.4 To eat a bowl of porridge it takes
a Scotsman 2 minutes;
an Englishman 4 minutes (less expert!);
a Welshman 8 minutes; and
an Irishman also 8 minutes.
If they all share a bowl of porridge (but have a spoon each) how long will it take all of them together? (We have to assume they don't slow each other down!)

Answer: 1 minute
Hint questions: "Will it take more or less than 8 minutes?" "More or less than 2 minutes?"
One way is to think about porridge-eating-rates.
$S = \frac{1}{2}$ a bowl per minute
$E = \frac{1}{4}$ of a bowl per minute
$W = I = \frac{1}{8}$ of a bowl per minute
So in 1 minute they finish off the whole bowl because $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{8}=1$.

1.10.5 A goat takes 3 minutes to eat a cabbage.
A rabbit takes 4 minutes and a mouse takes 5.
If a single cabbage were thrown to all three animals and they ate together, how long would they take between them to eat the whole cabbage?
What do you have to assume?

Answer: In 1 minute, they'd eat $\frac{1}{3}+\frac{1}{4}+\frac{1}{5}=\frac{47}{60}$ of the cabbage, so in $\frac{60}{47}=1\frac{13}{47}$ minutes, or 1 minute 17 seconds, they'd manage the whole thing.

That they don't fight or get in each other's way or try to eat each other!

1.10.6 Investigate A2, A3, A4, A5, A6 paper.
What is special about "A-size" paper?
Why does this happen?

Answer: Long side to short side is $\sqrt{2}:1$, so that cutting in half (halving the area) gives a piece of paper the same shape (sides in same ratio). This follows from saying that $x:1=1:\frac{1}{2}x$ so that $\frac{x}{1}=\frac{1}{\frac{1}{2}x}$ or $\frac{1}{2}x^2=1$, so that $x=\pm\sqrt{2}$, but only the positive solution makes sense here.

1.10.7 A script for a TV sit-com has about 6000 words per $\frac{1}{2}$ hour of running time. What is this in words per second?

Answer: about 3 (surprisingly, perhaps!) – not many pauses in dialogue during a TV sit-com!

1.10.8 Old-fashioned school maths questions; e.g., "If 200 men working 8 hours a day take 12 days to dig a trench 160 yards long, 6 yards wide and 4 yards deep, how many days will it take 90 men, working 10 hours a day to dig a trench 450 yards long, 4 yards wide and 3 yards deep?"

Answer: $12\times\frac{8}{10}\times\frac{200}{90}\times\frac{450}{160}\times\frac{4}{6}\times\frac{3}{4}=30$ days.

These are not too hard so long as you don't panic and just think about whether each factor will make the job take more time or less time.

1.10.9 Value for money for products from the supermarket (either £ per g or g per £).

Could visit a supermarket. (Or bring in empty packets or special offer adverts.)

1.10.10 If x people can pack 20 bags in x minutes, how long will it take $x+2$ people to pack 20 identical bags?

Answer is not $x+2$ minutes because it will take less time with more people.
Rate of working is $\frac{20}{x^2}$ bags per minute per person, so with $x+2$ people it will be, $\frac{20(x+2)}{x^2}$ bags per minute, so the time for 20 bags will be $20 \div \frac{20(x+2)}{x^2} = \frac{x^2}{x+2}$ minutes.
(Alternatively, it will take $\frac{x}{x+2}$ of the time it took x people, so that is $\frac{x}{x+2} \times x$ which is $\frac{x^2}{x+2}$ minutes.)

What assumption do you make?

We assume that with more people you get proportionately more work, and not just more chat!

1.10.11 If you have a bottle of drink worth £10 and another worth only £5, how much of each would you need to mix to make 1 bottle worth £8?

Answer: $\frac{3}{5}$ of the £10 bottle and $\frac{2}{5}$ of the £5 bottle, because $\frac{3}{5}\times 10+\frac{2}{5}\times 5=8$.
(Trial and improvement or solve simultaneously $10a+5b=8$ and $a+b=1$.)

1.10.12 A painter mistakenly mixes 5 litres of white paint with 3 litres of blue paint, when he meant to do it the other way round. How can he get the amount and colour that he wanted with the minimum wastage of paint?
(This would be more complicated if the different colours cost different amounts.)

Answer: He has used too much white paint, so he won't need to use any more of that. He needs to pour off $\frac{2}{5}$ of the mixture (= $\frac{2}{5}\times 8$ = 3.2 litres) leaving 3 litres of white and 1.8 litres of blue. Then add 3.2 litres of blue. This wastes an additional 3.2 litres only.

1.10.13 I have 50 cm^3 orange juice in one glass and 50 cm^3 water in another glass. I take 1 cm^3 of orange juice from the first glass and add it to the second. After stirring the second glass thoroughly I take 1 cm^3 of it and add it to the first glass (so both glasses again contain 50 cm^3). Is there more orange juice in the second glass or more water in the first?

Answer: The same, since the total volume in each glass is 50 cm^3, and it's either in one glass or the other: stirring thoroughly doesn't affect that.
In more detail, going stage by stage,

first glass		***second glass***	
orange	***water***	***orange***	***water***
50	0	0	50
49	0	1	50
$49\frac{1}{51}$	$\frac{50}{51}$	$\frac{50}{51}$	$49\frac{1}{51}$

So there is $\frac{50}{51}$ cm^3 of each in the other if thoroughly mixed.

1.10.14 A traffic warden takes 3 minutes 10 seconds to write out a parking ticket for a car. If he works at the same rate all afternoon, how long would it take him to make 60 similar bookings?

Answer: 3 hours, 10 minutes.
(Multiplying minutes and seconds by 60 gives hours and minutes.)

1.10.15 Gold. What does "24 carat" gold mean?
(Pupils could find out for homework, perhaps.)

1 carat = $\frac{1}{24}$ of the total mass, so 24 carat gold is 100% pure. For most purposes this is too soft, so copper or silver are commonly added.
18 carat gold is $\frac{18}{24}=\frac{3}{4}$ gold.

What about diamonds?
What's the maximum carat?

With diamonds, carats measure the mass of the diamond. 1 carat = 200 mg, so there's theoretically no limit to the number of carats.

1.10.16 Petrol.
A petrol pump delivers fuel at a rate of 25 litres per minute. How much money is it costing per second?
(Use current petrol price; e.g., 75 p per litre of unleaded fuel.)

Answer: With this figure, it comes to about 30 pence per second.

1.11 Fractions, Decimals and Percentages

- Fractions-decimals-percentages triangle. Can go clockwise or anticlockwise (see below). When going from fraction to %, if the denominator is a factor of 100, it's easiest simply to convert the fraction to something out of 100. But if the denominator isn't a factor of 100, then go round the triangle the other way *via* the decimal equivalent.
- The trickiest conversion is from a decimal into a fraction. The easiest way may be to write the decimal as a fraction "over 1" and then multiply numerator and denominator by 10 (or powers of 10) until both numbers are integers; then simplify as normal; e.g., 0.38 $=\frac{0.38}{1}=\frac{38}{100}=\frac{19}{50}$.

1.11.1 Three parallel number-lines can show the inter-conversion between fractions, decimals and % (go from 0 but beyond 1 to show that % can be greater than 100).

This reinforces that these are just three different ways of writing numbers.

1.11.2 **NEED** cards for "chain game" (see sheet of cards to use). Hand out all the cards (1 to each pupil, 2 or even 3 to some or do 1 or more yourself – you must use all the cards).
Locate the person who has the "start" card. He/she begins and reads out their card, and whoever has the answer reads theirs and so on.

Could time it – "Can we do it in less than $1\frac{1}{2}$ *minutes?" – although beware of this putting too much pressure on one or two individuals.*

(With a small class you may need to cover several cards yourself or make use of a teaching assistant to do that.)

1.11.3 Puzzle pictures (colour in the answers to produce a picture).

Often popular and available in books.

1.11.4 It's easy to make a table on the board with gaps for pupils to fill in (either on the board or on paper).

fraction	decimal	percentage
	0.7	
$\frac{4}{5}$		

Can throw in a recurring decimal as a challenge (see section 1.11.5 below).

The hardest ones are usually, e.g., converting 0.1% to 0.001 or 2 to 200%.

1.11.5 Converting recurring decimals to fractions; e.g., 0.4444…; 0.424242…; 0.123451234512345…; 0.007007007007…

Answers: $\frac{4}{9}$, $\frac{42}{99}$, $\frac{12345}{99999}$ *and* $\frac{7}{999}$.
You put the recurring part in the numerator and always as many 9's in the denominator as digits in the recurring section.
Of course, $\frac{3}{9}=\frac{1}{3}$ *(0.3333…), so that fits the pattern.*

Numbers greater than 1 are no more difficult; e.g., 1.1111111… or 1.23123123123123… or 1.123123123123123… or 3.555555555…

$1\frac{1}{9}=\frac{10}{9}$, $1\frac{231}{999}=\frac{1230}{999}$, $1\frac{123}{999}=\frac{1122}{999}$, $3\frac{5}{9}=\frac{32}{9}$.

Another method is to write, say, $0.2222...=\frac{2}{10}+\frac{2}{100}+\frac{2}{1000}+...$ *and to sum this as an infinite geometric series using* $\sum_{n=1}^{\infty}\frac{a}{x^n}=\frac{a}{x-1}$ *to give* $\frac{2}{10-1}=\frac{2}{9}$.

These can also be solved by letting x *be the fraction you're trying to find and writing, say,*
$x=0.7373737373...$
$100x=73.7373737373...$ *and subtracting gives*
$99x=73$ *so* $x=\frac{73}{99}$.

1.11.6 Investigate recurring digits in decimal expansions of fractions; e.g., which fractions have recurring decimals and which don't, how long the recurring unit is, etc.

Answers: denominators which take the form 2^x5^y where x and y are integers ≥ 0; e.g., halves, quarters, tenths, twentieths, etc. give terminating decimals.
If the denominator is n then the length of the recurring part of the decimal must be $\leq (n-1)$.
Beyond that it isn't easy to generalise, except that once the fraction is in its simplest form the length of the recurring unit depends only on the denominator and not on the numerator.

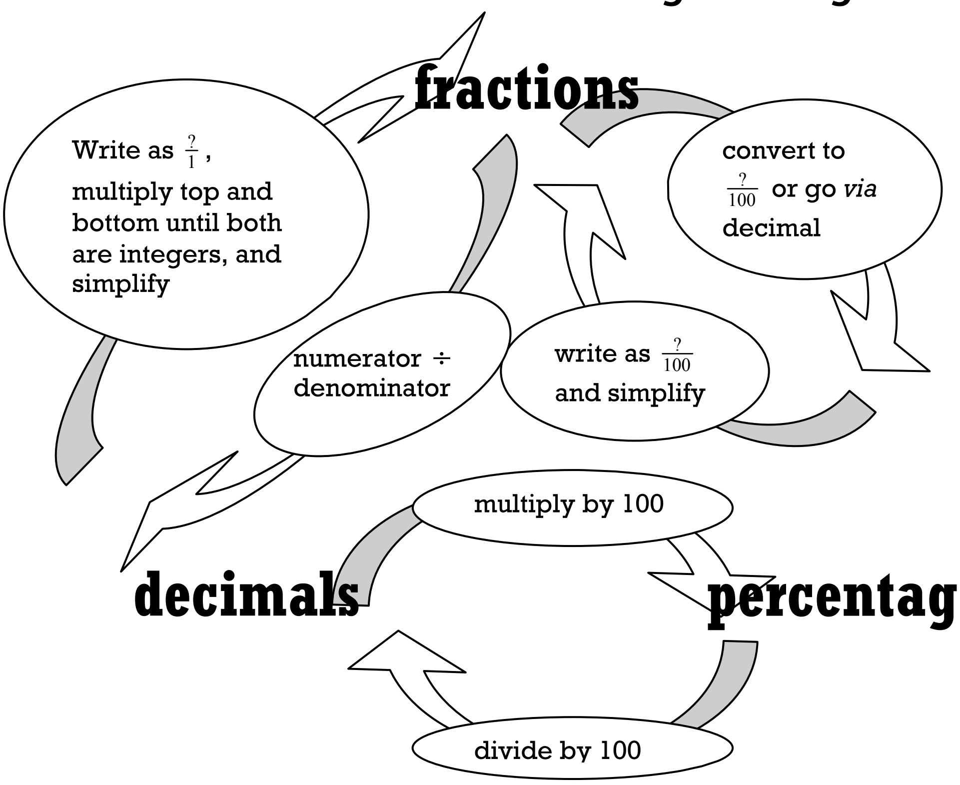
1.11.7 Comparing the size of similar fractions.
Using consecutive pairs of terms from the Fibonacci series (especially high terms) as the numerator and denominator makes fractions that are similar in size (approaching the golden ratio 1.618… as you take higher and higher terms).

e.g., the Fibonacci sequence begins 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, …
So $\frac{2}{3} > \frac{5}{8}$ and $\frac{8}{13} < \frac{13}{21}$, etc.

"Making the denominators the same" is a good method; in fact, although we may not think of it that way, that's what we're doing when we convert both of the fractions into decimals.
e.g., $\frac{8}{13} = \frac{8 \div 13}{13 \div 13} = \frac{0.6154}{1}$ (4 dp),
and $\frac{13}{21} = \frac{13 \div 21}{21 \div 21} = \frac{0.6190}{1}$ (4 dp).
Denominators are both 1 now.
(Converting to both to percentages is making the denominators both 100.)

But making numerators the same is just as good. For example, to compare $\frac{2}{7}$ and $\frac{4}{15}$, working out a common denominator of 7×15 is unnecessary. Instead we can make the first fraction into $\frac{4}{14}$ and then we can see that this is bigger than $\frac{4}{15}$, since $\frac{1}{14}$ is more than $\frac{1}{15}$, so four of them must be bigger than four of the other.

The Fractions-Decimals-Percentages Triangle



I am … start	I am … $\frac{7}{10}$	I am … 0.25	I am … 20%
You are … 0.7	You are … 25%	You are … $\frac{1}{5}$	You are … 90%
I am … $\frac{9}{10}$	I am … $\frac{3}{5}$	I am … 28%	I am … 0.75
You are … 0.6	You are … 0.28	You are … $\frac{3}{4}$	You are … 1%
I am … $\frac{1}{100}$	I am … 73%	I am … 0.1	I am … $33\frac{1}{3}\%$
You are … 0.73	You are … $\frac{1}{10}$	You are … $\frac{1}{3}$	You are … 0.41
I am … $\frac{41}{100}$	I am … 88%	I am … 0.81	I am … $\frac{2}{5}$
You are … 0.88	You are … $\frac{81}{100}$	You are … 40%	You are … 0.99

Photocopy onto coloured card (could laminate it) and guillotine along the **bold** lines only.
Both sheets together make a whole set – 32 cards should be enough for most classes.

I am … $\frac{99}{100}$	I am … 0.17	I am … 62%	I am … 0.04
You are … 17%	You are … $\frac{31}{50}$	You are … $\frac{1}{25}$	You are … 6%
I am … $\frac{3}{50}$	I am … 11%	I am … 0.35	I am … 0.05
You are … 0.11	You are … $\frac{7}{20}$	You are … 5%	You are … $\frac{3}{10}$
I am … 30%	I am … $\frac{37}{50}$	I am … $\frac{3}{25}$	I am … 9%
You are … 0.74	You are … 12%	You are … 0.09	You are … $\frac{13}{50}$
I am … 0.26	I am … 0.38	I am … 45%	I am … $\frac{1}{2}$
You are … 38%	You are … $\frac{9}{20}$	You are … 50%	You are … **stop**

Photocopy onto coloured card (could laminate it) and guillotine along the **bold** lines only.
Both sheets together make a whole set – 32 cards should be enough for most classes.

1.12 BIDMAS (Priority of Operations)

- **BIDMAS** may make more sense than BODMAS.

Brackets
Indices
Division **M**ultiplication
Addition **S**ubtraction

Indices means powers and roots.

Where 2 operations have the same level of priority we work from left to right in the sum.

The line in division behaves like a pair of brackets; e.g., $\frac{10-4}{2}=3$, not 1 or 8.

1.12.1 Write up something like $2+3\times4$ on the board. What is the answer?

How might different people give different answers to this?

Try it on a calculator to see what happens.

Answers: 20 or 14 depending on which operation you did first.
Introduce the word "operation".
In maths we don't like ambiguities – we want the same answer anywhere in the world, any time, no matter who does the question.
A scientific calculator gives 14; a basic calculator gives 20.

1.12.2 It's important that pupils can get the right answers using the bracket and memory keys on the calculator.
Put complicated combinations on the board.
What different answers do we get?
What does BIDMAS make it?
e.g., $\frac{25-(13-4)}{7-3\times2+1}\times\sqrt{\frac{8\times3^2}{11-(8+1)}}$

Fraction lines and square root symbols imply brackets around what they enclose.

Pupils can invent these.

Answer: 6 × 8 = 48

1.12.3 Setting out this work as clearly as possible will help – boxing (or colouring/highlighting) the portion you're working out and then writing the whole thing underneath with only the highlighted portion changed (into the answer) and then highlighting the next thing to do (see right).

Example: (keeping one = sign per line)

$$\frac{36-(7-3)^2}{4+2\times3}$$
$$=\frac{36-4^2}{4+2\times3}$$
$$=\frac{36-16}{4+2\times3}$$
$$=\frac{36-16}{4+6}$$
$$=\frac{20}{10}$$
$$=2$$

1.12.4 Puzzle pictures (colour in the answers to produce a picture).

Often popular and available in books.

1.12.5 True or false statements.
If false say what the correct value is and how the mistake could have been made.
e.g., 2 + 8 – 4 × 3 = 18

Pupils can invent these.

Answer: false, should be –2; the person has just gone left to right, ignoring BIDMAS.

It can be challenging but worthwhile to try to identify how errors (deliberate or not) have been made.

1.12.6 Without using any brackets, fill in $+-\times\div$ (as many of each as you like) to make these true.
9 9 9 9 9 = 89
9 9 9 9 9 = 11
9 9 9 9 9 = 108
1 1 1 1 1 = 3
Make up some for someone else.

Answers:
9 × 9 – 9 ÷ 9 + 9 = 89
9 ÷ 9 + 9 + 9 ÷ 9 = 11
9 + 9 × 9 + 9 + 9 = 108
1 + 1 × 1 + 1 × 1 = 3
and other possible solutions.

Can allow concatenation (sticking together adjacent digits; e.g., 4 8 makes 48) if you like.

Can use pupils' years of birth (e.g., 1 9 9 1).

As a challenge (e.g., for homework) make ten totals from the year of your birth. You're not allowed to alter the order of the digits, but you can use $+-\times\div!\sqrt{\ }$, powers, concatenation and brackets (at your discretion). Must obey BIDMAS!

Can restrict to integer answers or not.

1.12.7 By putting in brackets (as many pairs as you like) what different answers can you make to this? (You mustn't change anything else.)
4 × 5 – 2 + 3 × 6 ÷ 2

4 × 5 – 2 + 3 × 6 ÷ 2 = 27
4 × (5 – 2) + 3 × 6 ÷ 2 = 21
4 × (5 – 2 + 3) × 6 ÷ 2 = 72
4 × (5 – 2 + 3 × 6) ÷ 2 = 42
4 × (5 – 2 + 3 × 6 ÷ 2) = 48
… and lots more!

1.12.8 BIDMAS applies to algebra; e.g., $E = mc^2$ isn't $(mc)^2$ and when we write $3x+4$ we assume multiplication of 3 and x before the addition of the 4.

A common error when calculating the area of a circle is to multiply π *by the radius and then square the answer. For this reason,* $r^2\pi$ *is sometimes more successful than* πr^2 *.*

1.12.9 **NEED** "Insert the signs" sheets.

Quite difficult.

1.12.10 Four Fours.
Given a maximum of four number 4's and as many + – × ÷ signs as you want, can you make all the numbers from 1 to 20?

Which numbers can be made in more than one way?

Start by trying to make 1 (= 4 ÷ 4).
Could allow $\sqrt{\ }$ *sign (hence 2) or ! (factorial).*

1 = 4 ÷ 4	*11 = 44 ÷ 4* *if allowed*
2 = 4 ÷ 4 + 4 ÷ 4 *2 = (4+4) ÷ 4* *2 =* $\sqrt{4}$	*12=* $4\sqrt{4}$ *+ 4* *12 = 4 + 4 + 4* *12 = 4 × 4 – 4*
3 = 4 – 4 ÷ 4	*13 = 44 ÷ 4 +* $\sqrt{4}$
4	*14 = 4 × 4 –* $\sqrt{4}$
5 = 4 + 4 ÷ 4	*15 = 4 × 4 – 4 ÷ 4*
6 = 4 + (4+4) ÷ 4 *6 = 4 +* $\sqrt{4}$	*16 = 4 × 4* *16 = 4 + 4 + 4 + 4* *16 =* 4^4 *÷ (4 × 4)*
7 = 4 + 4 – 4 ÷ 4	*17 = 4 × 4 + 4 ÷ 4*
8 = 4 + 4 *8 =* $4\sqrt{4}$	*18 = 4 × 4 +* $\sqrt{4}$
9 = 4 + 4 + 4 ÷ 4	*19 = 4 × 4 +* $\sqrt{4}$ *+ 4 ÷ 4,* *but* ***uses five 4's***
10 = 4 + 4 + 4 – $\sqrt{4}$ *10 = 4 + 4 +* $\sqrt{4}$	*20 = 4 × 4 + 4* *20 = 4 × (4 + 4 ÷ 4)* *20 = 4! – 4*

You can make 19 from four 4's if you do, e.g.,
4 ÷ 4 + !4 + !4
$!4 = 9$ *, where* $!n$ *is the "subfactorial of* n *", the number of ways of arranging* n *items so that each one is in the "wrong" place (e.g., the number of ways of putting 4 letters into 4 envelopes so that no letter is in the correct envelope). But if no-one knows about this notation, this might count as "cheating"!*
(See section 3.4.5.)

Another "cheat" for 19 is $4! - \frac{\sqrt{4}}{.4}$ *(note the decimal point in the denominator)!*

… and many other possibilities …

Insert the Signs

Insert + – × ÷ and ( ) signs in between the numbers to make the sums correct.

e.g., 6 6 3 = 8 insert + and ÷
6 + 6 ÷ 3
= 6 + 2
= 8

You are also allowed to put digits next to each other to make numbers (concatenation) (e.g., 1 2 can become 12), but you can't alter the *order* of the digits.

e.g., 1 2 5 2 = 36 insert × and –
12 × (5 – 2) = 36

Try these.

1 4 1 = 3	5 3 6 7 = 1
4 2 7 = 6	8 7 5 5 = 4
5 6 2 = 9	6 4 2 2 = 9
8 2 8 = 12	3 9 1 7 = 16
6 4 9 = 15	9 3 9 4 = 25
8 4 6 = 18	2 7 4 3 = 36
5 1 6 = 21	4 9 4 2 = 49
9 6 4 = 24	6 4 5 8 = 64
7 4 9 = 27	1 8 5 2 = 81
2 4 7 = 30	6 8 5 2 = 100

1 5 5 4 = 101	8 4 9 7 = 0
2 3 5 6 = 102	6 8 4 6 = 11
6 9 2 5 = 103	5 7 9 3 = 22
2 6 7 8 = 104	1 4 9 3 = 33
8 1 2 9 = 105	3 5 2 8 = 44
9 5 6 7 = 106	1 6 2 8 = 55
9 6 8 7 = 107	2 7 4 3 = 66
6 7 4 7 = 108	3 7 4 7 = 77
8 5 6 9 = 109	4 1 4 8 = 88
5 5 7 3 = 110	8 1 3 5 = 99

Insert the Signs **ANSWERS**

Insert + – × ÷ and ( ) signs in between the numbers to make the sums correct.

e.g., 6 6 3 = 8 insert + and ÷

$6 + 6 \div 3$
$= 6 + 2$
$= 8$

You are also allowed to put digits next to each other to make numbers (concatenation) (e.g., 1 2 can become 12), but you can't alter the *order* of the digits.

e.g., 1 2 5 2 = 36 insert × and –

$12 \times (5 - 2) = 36$

Try these.

1 × 4 – 1	= 3	**5 – 3 + 6 – 7**	= 1
42 ÷ 7	= 6	**(8 + 7 + 5) ÷ 5**	= 4
5 + 6 – 2	= 9	**6 + 4 – 2 ÷ 2**	= 9
8 ÷ 2 + 8	= 12	**3 + 91 ÷ 7**	= 16
6 × 4 – 9	= 15	**9 + 3 + 9 + 4**	= 25
8 + 4 + 6	= 18	**27 × 4 ÷ 3**	= 36
5 + 16	= 21	**(4 + 94) ÷ 2**	= 49
96 ÷ 4	= 24	**6 × 4 + 5 × 8**	= 64
(7 – 4) × 9	= 27	**1 + 8 × 5 × 2**	= 81
2 + 4 × 7	= 30	**6 × 8 + 52**	= 100

1 + 5 × 5 × 4	= 101	**8 ÷ 4 – 9 + 7**	= 0
(2 + 3 × 5) × 6	= 102	**68 ÷ 4 – 6**	= 11
6 × 9 × 2 – 5	= 103	**(57 + 9) ÷ 3**	= 22
26 + 78	= 104	**1 × 4 × 9 – 3**	= 33
8 × 12 + 9	= 105	**352 ÷ 8**	= 44
9 × (5 + 6) + 7	= 106	**1 + 62 – 8**	= 55
9 + (6 + 8) × 7	= 107	**2 × (7 + 4) × 3**	= 66
6 × (7 + 4 + 7)	= 108	**3 × 7 × 4 – 7**	= 77
8 × 5 + 69	= 109	**4 × (14 + 8)**	= 88
5 + 5 × 7 × 3	= 110	**8 × 13 – 5**	= 99

1.13 Negative Numbers

- Temperature is an obvious place to start. A temperature scale/thermometer drawing (in °C) from 30 to –10 (1° per 0.5 cm) down the margin on the left side of a page in an exercise book can be very useful.
  It may be worth trying to say "higher" and "lower" rather than "bigger" and "smaller"; > and < will need explaining here. Some people prefer to say "negative 3" rather than "minus 3", saving "minus" for operations ("negative" is an adjective; "minus" is a verb), although no-one does this with temperature.
- Adding and subtracting positive numbers (getting this much hotter or colder) is a good way in to negative numbers as something inevitable if we take away too much! Adding negative numbers may be thought of as pouring cold liquid into a bath, reducing the overall temperature.
- The difference between two temperatures can lead to subtracting negative numbers; e.g., $3-3=0$, $3-2=1$, $3-1=2$, $3-0=3$, so logically $3--1=4$, and this corresponds to 3 and –1 being 4 apart on the temperature scale / number-line.
- An alternative scheme is to say that + means "followed by", so that $3-2$ means $3+-2$ ("3 followed by –2"), $3-+2$ becomes $3+-2$ (adding the inverse is what subtracting means), and $3--2$ becomes $3++2$ since the inverse of –2 is +2 (and adding the inverse is what subtracting means).
- Multiplying can be introduced by noting that $3\times-2=-2+-2+-2=-6$.
  (If $3\times2=6$ then 3×-2 can't be the same.)
- For dividing, $-6\div3$ works by dividing the line joining –6 to 0 on the number-line into 3 equal pieces; $6\div-3$ is much harder conceptually and probably has to be seen as sensible by looking at number patterns (see below).
- To square a negative number on scientific calculators you need to use brackets: $(-3)^2=9$, whereas $-3^2=-9$.
- Pupils may need to get out of the habit of using a "dash" in ways that could be ambiguous; e.g., "answer – 17" (17 or –17?). "In maths, use maths signs only if you mean the maths meaning!"

1.13.1 Tell me a situation where negative numbers would make no sense at all, or a situation where they would make some sense.

Could draw up a table on the board of "sensible" versus "not sensible".

1.13.2 True or false statements involving < and >.

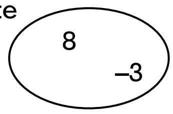
Easy to make up.

"Confidence". Instead of writing down "T" or "F", an alternative is for pupils to hold their arms out horizontally above the desk as you make the true or false statements. For a true statement, they wiggle their fingers; for a false statement they bang their palms down on the desk.

It can be an easy measure of how confident pupils are about negative numbers! Can do it with eyes closed if there's too much "peer pressure" effect!

Alternatively, you can simply do "stand up" (true) or "sit down" (false).

1.13.3 Write



on the board.
Altogether I've got 5.
Take away the –3 (rub it out) – what's left? 8.
So $5--3=8$

This approach may be helpful for some pupils.

Cancelling a debt you owe makes you better off. Maybe there is a school system of rewards and punishments where taking off "penalty points" is equivalent to gaining "house points"?

1.13.4 Puzzle pictures (colour in the answers to produce a picture).

Often popular and available in books.

1.13.5 Discussion: "Two minuses make a plus" – when does that work and when doesn't it?

On the whole do you think it's a helpful rule? Would you use it if teaching a younger brother/sister?

Most rules have a limited range of situations in which they're appropriate; e.g., scientific laws.

Answer:
It works only when

1. *the signs are next to each other;*
   e.g., 3 – – 4 = 3 + 4 = 7;
2. *multiplying or dividing;*
   e.g., – 3 × – 4 = 12 and – 30 ÷ – 10 = 3.

It doesn't work in any other situation;
e.g., it doesn't apply to – 3 – 4 = – 7.
(If it's – 3˚C and it gets 4˚C colder, it goes down to – 7˚C.)

1.13.6 Negative co-ordinates are a useful context.

1. If three vertices of a rectangle are (–3,1), (2,1), (2, –2) where is the 4th vertex? What are the perimeter and the area?
2. If 3 points are (1,3), (–2,2) and (1,–2) what 4th point will make a) a kite; b) a parallelogram; c) an arrowhead?

Answers:

1. *(–3,–2), perimeter = 16 units,*
   area = 20 sq units
2. *many possible answers*

1.13.7 Magic squares with negative numbers (see sheet). (Photocopy the sheet or write them onto the board.)
You may need to emphasise that it's the *total* (sum) along each row/column/diagonal that matters.

These can be quite tricky, but because the same numbers have to be added in different directions, there is a natural checking process going on. (They shouldn't need "marking" afterwards.)

1.13.8 **NEED** cards with positive and negative numbers (one number on each card). (You can make these fairly easily.)
Quiz with 2 teams: Each correct answer means that the next card is turned over and added to the team's score (so the score goes down if a negative number is on the card).

You can reward a particularly good answer by letting the team remove any card they choose (subtracting a negative number).

You can use fractions to make it more difficult.

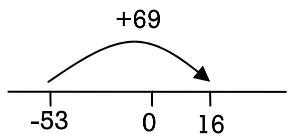
The quiz itself could be mathematical or not.

1.13.9 If $x + y = -8$, what could x and y be?

Many possible answers; you could sketch the graph to show the infinity of pairs of points.

1.13.10 Addition and subtraction can be much harder with larger values; e.g., –53 + 69

Answer: 16
You can use an "empty" number-line; e.g.,



1.13.11 Get used to writing any "sum" in 3 ways.
e.g., 37 – 15 = 22 is equivalent to 22 + 15 = 37 and 37 – 22 = 15.
So if we're convinced that 10 = 18 + – 8 it follows that 10 – 18 = –8 and 10 – – 8 = 18.
(The last one is "ten minus negative eight".)

This could lead to rearranging formulas such as $x = \pm a \pm b$.

1.13.12 A window-cleaner is standing on the *middle step* of his ladder. As he works, he climbs up 4 steps. Then he climbs down 7 steps. Finally he climbs up 10 steps to the top of the ladder. How many steps are there on the ladder?

Answer: 14.
+4 – 7 + 10 = 7 from the middle to the top, so there must be 14 from the bottom to the top.

1.13.13 **NEED** "Multiplying and Dividing Negative Numbers" sheet.

See "Teacher's Notes" on the sheet.

Magic Squares (Negative Numbers)

Fill in the numbers that are missing in these magic squares.
In each square the total along every column, every row and both diagonals is the same.
Different squares have different totals. Some of the numbers will be negative.

1

–2		
–3	–1	
2		

2

	–23	
	–19	
–22	–15	

3

–4		
	–2	
	–10	0

4

		–8
	–9	
–10		–6

5

	–3	–10
		–5
		–6

6

–3	–6	9
		3

7

0		4
	1	
		2

8

		–3
–6	–4	
–5		

9

	–31	
	–21	
	–11	–16

10

–8		
–4	–9	–2

11

5.5	0.5	1.5
3.5		

12

5.5		
–4.5	–0.5	
–2.5		

Magic Squares (Negative Numbers) ANSWERS

Fill in the numbers that are missing in these magic squares.
In each square the total along every column, every row and both diagonals is the same.
Different squares have different totals. Some of the numbers will be negative.

1 magic total –3

–2	**3**	**–4**
–3	–1	**1**
2	**–5**	**0**

2 magic total –57

–18	–23	**–16**
–17	–19	**–21**
–22	–15	**–20**

3 magic total –6

–4	**6**	**–8**
–6	–2	**2**
4	–10	0

4 magic total –27

–12	**–7**	–8
–5	–9	**–13**
–10	**–11**	–6

5 magic total –21

–8	–3	–10
–9	**–7**	–5
–4	**–11**	–6

6 magic total 0

–3	–6	9
–12	**0**	**–12**
–9	**6**	3

7 magic total 3

0	**–1**	4
5	1	**–3**
–2	**3**	2

8 magic total –12

–1	**–8**	–3
–6	–4	**–2**
–5	**0**	**–7**

9 magic total –63

–26	–31	**–6**
–1	–21	**–41**
–36	–11	–16

10 magic total –15

–8	**–1**	**–6**
–3	**–5**	**–7**
–4	–9	–2

11 magic total 7.5

5.5	0.5	1.5
–1.5	**2.5**	**6.5**
3.5	**4.5**	**–0.5**

12 magic total –1.5

5.5	**–8.5**	**1.5**
–4.5	–0.5	**3.5**
–2.5	**7.5**	**–6.5**

Multiplying and Dividing Negative Numbers

1. Fill in the shaded boxes in the multiplication square below.
2. Look at the patterns along the rows and columns and continue those patterns into the un-shaded boxes.
3. Complete the whole table.

5											
4											
3											
2					–2	0	2	4	6	8	10
1											
0											
–1											
–2											
–3											
–4											
–5											
	–5	**–4**	**–3**	**–2**	**–1**	**0**	**1**	**2**	**3**	**4**	**5**

Use your table to say what happens when you
1. multiply a positive number by a negative number;
2. multiply together two negative numbers.

What does the table tell you about *dividing*?

Multiplying and Dividing Negative Numbers

1. Fill in the shaded boxes in the multiplication square below.
2. Look at the patterns along the rows and columns and continue those patterns into the un-shaded boxes.
3. Complete the whole table.

5											
4											
3											
2					–2	0	2	4	6	8	10
1											
0											
–1											
–2											
–3											
–4											
–5											
	–5	**–4**	**–3**	**–2**	**–1**	**0**	**1**	**2**	**3**	**4**	**5**

Use your table to say what happens when you
1. multiply a positive number by a negative number;
2. multiply together two negative numbers.

What does the table tell you about *dividing*?

Multiplying and Dividing Negative Numbers

1. Fill in the shaded boxes in the multiplication square below.
2. Look at the patterns along the rows and columns and continue those patterns into the un-shaded boxes.
3. Complete the whole table.

5	-25	-20	-15	-10	-5	0	5	10	15	20	25
4	-20	-16	-12	-8	-4	0	4	8	12	16	20
3	-15	-12	-9	-6	-3	0	3	6	9	12	15
2	-10	-8	-6	-4	-2	0	2	4	6	8	10
1	-5	-4	-3	-2	-1	0	1	2	3	4	5
0	0	0	0	0	0	0	0	0	0	0	0
-1	5	4	3	2	1	0	-1	-2	-3	-4	-5
-2	10	8	6	4	2	0	-2	-4	-6	-8	-10
-3	15	12	9	6	3	0	-3	-6	-9	-12	-15
-4	20	16	12	8	4	0	-4	-8	-12	-16	-20
-5	25	20	15	10	5	0	-5	-10	-15	-20	-25
	-5	**-4**	**-3**	**-2**	**-1**	**0**	**1**	**2**	**3**	**4**	**5**

Use your table to say what happens when you
1. multiply a positive number by a negative number;
2. multiply together two negative numbers.

What does the table tell you about *dividing*?

TEACHER'S NOTES AND ANSWERS

Once the shaded squares are filled in, you can start with the $1 \times$ row and count back 5, 4, 3, 2, 1, 0, … What comes next?

Pupils can complete the rows and columns in this way.

This still leaves the bottom left corner. The patterns in the columns above and the rows to the right help pupils to see that the numbers in this bottom left corner must be positive.

The aim is for pupils to see that it makes sense to define multiplication and division of *negative* numbers as below because then everything fits in with the patterns we get when we multiply and divide *positive* numbers.
It is reasonable to accept that "in-between" numbers (decimals and fractions) will work in the same kind of way.

So, for multiplying or dividing two numbers,

2nd number \ 1st number	+	–
+	+	–
–	–	+

Extra Task Make up ten multiplications and ten divisions each giving an answer of –8. (e.g., $-2 \times -2 \times -2$ or -1×8, etc.)

1.14 Indices

- The rules of indices can be introduced with powers of ten. 10^3 is 10 × 10 × 10 and 10^5 is 10 × 10 × 10 × 10 ×10, so multiplying these gives 10 × 10 × 10 × 10 × 10 × 10 × 10 × 10 = 10^8. Multiplying an integer by 10 means adding a 0 on to the right end, so adding 1 to the power of 10 makes sense. The division rule comes from cancelling tens from the top and bottom of a fraction, and when the denominator is larger than the numerator we're in to negative indices.

  Alternatively, multiplying by 10^a moves all the digits a places to the left, and dividing by 10^b moves all the digits b places to the right, so the overall effect is a move $a-b$ places to the right (or $b-a$ places to the left); so $10^a \div 10^b = 10^{a-b}$.
- The trickiest rule is $x^{\frac{a}{b}} = \sqrt[b]{x^a}$, and this is probably better done (especially initially) in two stages, so e.g., $8^{\frac{2}{3}} = (8^{\frac{1}{3}})^2 = (\sqrt[3]{8})^2 = 2^2 = 4$.

  Confident calculator use is very important, using the x^y or y^x or ^ buttons for powers, but the EXP button for standard form (and *not* typing in the 10 or the ×); i.e., for 6×10^4 typing "6 EXP 4" only.

1.14.1 "Sarah, give me a number between 1 and 10."
Sarah says (say) 6; I write 36 on the board.
Repeat a couple of times. "What's going on?"

Treating the teacher as a function machine.
Can use for "cubing", etc.
A good way of getting pupils to express in words what a function is doing to a number.

1.14.2 What's the difference between 3^4 and 4^3?
Which do you think is bigger?
Can you think of cases where the answers are the same? i.e., $a^b = b^a$

Explain the difference between 3^4 and 4 × 3.

3^4 = 3 × 3 × 3 × 3, and 4 × 3 = 3 + 3 + 3 + 3
(4 × 3 = 3 × 4 = 4 + 4 + 4)

3^4 = 3 × 3 × 3 × 3 = 81; 4^3 = 4 × 4 × 4 = 64

Obviously, there are solutions if $a = b \neq 0$ (0^0 is undefined); e.g., $3^3 = 3^3$.
There are also infinitely many solutions where $a \neq b$: one value is always between 1 and e (2.71828…) and the other is $> e$. The only integer solution where $a \neq b$ is $2^4 = 4^2$.

1.14.3 Calculating square roots by hand may be of interest to some pupils.

See old-fashioned textbooks or the internet.

1.14.4 Work out 4^1, 4^2, 4^3 and 4^4.
What are the units digits for each one?
What do you notice?
What would the last digit be in 4^{20}?
What about 4^{123}?

Answers: 4, 6, 4, 6
The units digit in 4^n is 4 when n is odd and 6 when n is even. This works only for positive integer values of n. For $n = 20$, the last digit will be 6; for $n = 123$ it will be 4.

1.14.5 Are these always true, sometimes true (if so say when) or never true?
(Try it with numbers to see.)

1. $(a+b)^2 = a^2 + b^2$
2. $(a-b)^2 = a^2 - b^2$
3. $(ab)^2 = a^2b^2$
4. $(\frac{a}{b})^2 = \frac{a^2}{b^2}$
5. $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$
6. $\sqrt{a-b} = \sqrt{a} - \sqrt{b}$
7. $\sqrt{ab} = \sqrt{a}\sqrt{b}$
8. $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

Answers: ("iff" means "if and only if")

1. *true iff a or b = 0*
2. *true iff b = 0 or $a = b$*
3. *true always*
4. *true unless $b = 0$*
5. *true iff a or b = 0*
6. *true if b = 0 and $a > 0$ or $a = b > 0$*
7. *true unless a or b < 0*
8. *true unless a < 0 or b ≤ 0*

1.14.6	Find two consecutive integers whose squares differ by 31. Find two consecutive odd numbers whose squares differ by 80.	*Answer: 15 and 16* *Answer: 19 and 21*
1.14.7	Sum consecutive square numbers and try to verify that $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$. Could also try $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1) = \left(\sum_{r=1}^{n} r\right)^2$.	*And even the sums of 4th and 5th powers:* $\sum_{r=1}^{n} r^4 = \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$ $\sum_{r=1}^{n} r^5 = \frac{1}{12}n^2(n+1)^2(2n^2+2n-1)$
1.14.8	There are 20 integers less than 30 that can be written as the difference of two squares. What are they?	*3, 5, 7, 8, 9, 11, 12, 13, 15 (2 ways), 16, 17, 19, 20, 21 (2 ways), 23, 24 (2 ways), 25, 27 (2 ways), 28, 29* *The only ones that won't work are 2 × an odd number.*
1.14.9	Square and square root as inverses. I think of a number and square it. I get 36. The number wasn't six. What was the number? Does this happen with cubes as well? Could try to find $\sqrt{-36}$. Why does the calculator say "ERROR"?	*Answer: –6* *Answer: No, because* $-6 \times -6 \times -6 = -216$. *There's no "real number answer", but there are so-called "imaginary" or "complex" numbers. The answer to this would be* $6i$*, where* $i = \sqrt{-1}$.
1.14.10	Happy Numbers "Take a 2-digit number. Square each digit and add up the numbers you get (you could call it the "square digit sum"). Keep going until you have a good reason to stop." If you hit a 3-digit number, just square each digit and add the three numbers you get, and carry on.	*It's called a "Happy Number" if you end up at 1 eventually when you do this.* *You can make a "map" of which numbers lead to which numbers and look for loops and branches.* *Happy numbers include 10, 100, 1000 (1 step away from 1); 13, 31, 130, 86, 68 (2 steps away from 1) and 28, 82, 32, 23, 79, 97 (3 steps away from 1).*
1.14.11	Squares on a Chessboard. How many squares are there on a chessboard? There are more than 64. Start small (i.e., a 1 × 1 board has just 1 square). Be systematic. Extension: How many rectangles are there? (A square counts as a rectangle.)	*Answer: 204 (fewer than you might have guessed); the sum of the 1st eight square numbers. Count all of size 1 × 1 (8 × 8 = 64), size 2 × 2 (7 × 7 = 49), etc., separately and add them up. Or find the total for different sized boards (1, 5, 14, 30, 55, 91, 140, 204).* *Answer: 1296 (the sum of the 1st eight cube numbers) (see section 1.17.2).*
1.14.12	Towers of Hanoi. Ancient puzzle: You have 3 towers, n discs stacked on one of them in order of size (biggest at the bottom). You have to get the whole stack onto another tower in the same order (biggest at the bottom). You can only move one disc at a time, and at no point are you allowed to have a larger disc on top of a smaller disc. What is the minimum number of moves possible? $m_n = 2m_{n-1} + 1$ *is an inductive formula.*	*Answer:* $2^n - 1$*, a big number if you have more than a few discs.* *One way to think of it is mentally to split up a stack of* n *discs into the one on the bottom and the stack of* $n-1$ *discs above it. If* m_{n-1} *is the number of moves for* $n-1$ *discs, then you just have to move them onto the middle stack and then move the one remaining disc onto the third stack. Then move the* $n-1$ *discs again onto the third stack (another* m_{n-1} *moves) on top of the biggest disc and you're done.* *Therefore* $m_n = m_{n-1} + 1 + m_{n-1} = 2m_{n-1} + 1$. *Since you're doubling (and adding one) each time, the formula is bound to involve* 2^n.

1.14.13 Sweets (or grains of rice on a chessboard). "In the old days in maths lessons, pupils used to sit with the person who was the best right at the front, then the next best person next to them, and so on till the person who was the worst right at the back. (I don't know how they did it because Katie might be better than David at one bit of maths and David might be better than Katie at another bit of maths, so it can't have been easy.) Imagine we were sat like that today, and I decided to give out sweets. David (at the back) gets 1 sweet (to console him), Jenny gets 2 sweets, Henry gets 4 sweets, and so on, doubling each time.
I have two questions.
First, how many sweets will Katie (in number 1 position) get, and second, how many sweets would I need to buy to do this."

Careful who's sitting where before you do this!

It's surprising how much pupils will enjoy an imaginary and unfair distribution of sweets!

Imagine a class of size 10 and see what happens. Pupils may think "proportionally" (e.g., "work it out for the first 4 pupils and times it by 8 for all 32 pupils"), but it doesn't work.

"Katie" will get 2^{n-1} *sweets, where* n *is the class size today; e.g., for* $n = 30$, $2^{29} = 536,870,912$.

Total number of sweets = $\sum_{i=1}^{n} 2^{i-1} = 2^n - 1$.
So each pupil gets 1 more than the total of everyone up to them.
So for a class of 30 I'd need 1,073,741,823 sweets (over a billion). If they were penny sweets, it would cost me £10,737,418.23.

How many years of my salary (say £20 000 per annum) would it cost?

About 500 years (assuming I spent no money on anything else)!

1.14.14 Tournaments.
Imagine we all want to play in a table tennis tournament. What if there were 32 people here?

Answers:
Works nicely with a power of 2: 16 matches in 1st round, 8 in 2nd, 4 in 3rd, 2 semi-finals and 1 final (draw a tree diagram).

How many matches would there be?

31 (= 32 – 1) because $\sum_{i=1}^{n} 2^{i-1} = 2^n - 1$.

What if Edward didn't want to play and we couldn't find anyone else?
Would we have to cancel the whole thing?

No – someone could join in at round 2 and then it would be OK.
The number of matches is still 1 less than the number of people playing because if n *people take part then* $n-1$ *people must lose once each (the other person wins the tournament). Each match has one winner and one loser, so there must be* $n-1$ *matches.*

1.14.15 I have 2 parents, 4 grandparents, 8 great-grandparents, and so on. If I count parents as 1 generation back, grandparents as 2 generations back, etc., then how many great-great-etc.-grandparents do I have 20 generations back?
Estimate how long ago that would have been.

Answer: 2^{20} *= 1 048 576 people 20 generations back, and taking 25 years as an average "generation gap" this would have been 20 × 25 = 500 years ago (Elizabethan times).*

What about 30 generations back?

Again, 2^{30} *= about 1 billion ancestors 30 × 25 = 750 years ago (medieval times).*

What about 33 generations back?

Now 2^{33} *= 8.6 billion, more than the total number of people in the world today, and there were far fewer then, and these are just* my *ancestors! What about everyone else's?*

It's estimated that 1000 years ago there were only about 300 million people in the world in total! If everyone is descended from "Adam and Eve", then we must share a lot of distant ancestors.

The answer is that we're counting the same people more than once because some of my ancestors will have married people with some of the same ancestors.

1.14.16	Knights of the Round Table (see sheet; answers below).	*Interesting problem involving powers of 2.*
1.14.17	Why do the square numbers alternate odd, even, odd, even, …?	*Answer: Odd number × odd number = odd number, and even × even = even.* *So because the positive integers alternate odd, even, odd, even, so do the square numbers.*
1.14.18	Exponential Curves (growth or decay). e.g., bacteria growth curves (numbers of bacteria against time); rabbit population in Australia (numbers of rabbits against time); radioactive decay curves (activity against time), cooling curves (temperature against time).	*An opportunity for some mathematical modelling and discussing of assumptions.* *In each case, the rate of increase or decrease is roughly proportional to the current value.*
1.14.19	The number of bacteria in my fridge doubles every week. If there were 20 000 at the end of the 12th week, after how many weeks were there 10 000 bacteria?	*Answer: 11 weeks (obvious when you think about it).*
1.14.20	Which do you think is larger: $2\sqrt{\frac{2}{3}}$ or $\sqrt{2\frac{2}{3}}$? Find another instance of this.	*Answer: they're equal; easily seen by squaring the left hand side to give* $4\times\frac{2}{3}=2\frac{2}{3}$. *Other examples are* $3\sqrt{\frac{3}{8}}=\sqrt{3\frac{3}{8}}$, $4\sqrt{\frac{4}{15}}=\sqrt{4\frac{4}{15}}$, $5\sqrt{\frac{5}{24}}=\sqrt{5\frac{5}{24}}$, *etc.*
1.14.21	Logarithms. Pupils could experiment with the mysterious $\log$ button on the calculator to try to work out what it does. Pupils could be asked to work out $\log 1000$, $\log 100$, $\log 1$, $\log 0.01$ and to try to relate the answers to the numbers that were "logged". Why do $\log 0$ and $\log -10$ give ERROR?	*Answers: 3, 2, 0, –2* *Logarithms are just powers; the logarithm of a number is the power that 10 would have to be raised to to get that number, so because* $10^{-2}=0.01$ *we can say that* $-2=\log 0.01$. *(Logarithms can be to* bases *other than 10.)* *The equations* $10^x=0$ *and* $10^x=-10$ *have no real solutions.*

Knights of the Round Table ANSWERS

The only safe position is 73 places from the left of the King.

See the table on the sheet for the best position p for different numbers of knights n. Positions are counted clockwise around the table, counting as 1 the knight immediately to the King's left. Every time n reaches a power of 2 (e.g., $16 = 2^4$), the winning position goes back to $p=1$, and from then onwards the presence of every additional knight moves the optimum position two knights further on.

So if $m=$ the biggest power of 2 that is less than n, then $p=2(n-m)+1$.

e.g., if $n=100$, then $m=2^6=64$, and therefore $p=2(100-64)+1=73$.

So for 100 knights the 73rd position is the best.

And if $n=1000$, $m=2^9=512$, and therefore $p=2(1000-512)+1=977$.

So for 1000 knights the best position is the 977th.

Knights of the Round Table

Once upon a time there was a King who had a round table.

He wanted his daughter to get married, and he invented an unusual way of deciding which of his knights she would marry.

He invited the knights to dinner and sat them at the round table.

After dinner the King took out his sword, and passing by the knight immediately to his left, cut off the head of the knight next to him.

He then went on around the table cutting off the head of every other knight.

He kept going – missing himself out, of course! – until there was just one knight left.

This was the knight who would survive to marry the King's daughter.

So if 100 knights are invited, where is the best place to sit?

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Knights of the Round Table

n = number of knights at the table   ***p*** = position of the knight who wins

n	*p*	*n*	*p*	*n*	*p*	*n*	*p*
1	1	26	21	51	39	76	25
2	1	27	23	52	41	77	27
3	3	28	25	53	43	78	29
4	1	29	27	54	45	79	31
5	3	30	29	55	47	80	33
6	5	31	31	56	49	81	35
7	7	32	1	57	51	82	37
8	1	33	3	58	53	83	39
9	3	34	5	59	55	84	41
10	5	35	7	60	57	85	43
11	7	36	9	61	59	86	45
12	9	37	11	62	61	87	47
13	11	38	13	63	63	88	49
14	13	39	15	64	1	89	51
15	15	40	17	65	3	90	53
16	1	41	19	66	5	91	55
17	3	42	21	67	7	92	57
18	5	43	23	68	9	93	59
19	7	44	25	69	11	94	61
20	9	45	27	70	13	95	63
21	11	46	29	71	15	96	65
22	13	47	31	72	17	97	67
23	15	48	33	73	19	98	69
24	17	49	35	74	21	99	71
25	19	50	37	75	23	100	73

1.15 Standard Form

- Confident calculator use is very important, using the x^y or y^x or ^ buttons for powers, but the EXP button for standard form (and *not* typing in the 10 or the ×); i.e., for 6×10^4 typing "6 EXP 4" only.
- One approach to positive and negative indices is to say that $\times10^5$ means multiply by 10 "five times" (× 10 × 10 × 10 × 10 × 10), whereas $\times10^{-5}$ means divide by 10 "five times" (÷ 10 ÷ 10 ÷ 10 ÷ 10 ÷ 10).

1.15.1 Science provides nice contexts for very big and very small numbers; could look at the prefixes used (see sheet).

Large: space, galaxies, etc.
Small: atoms, molecules, etc.
(see sheet)

1.15.2 Standard Form. You can introduce this by writing the mass of the earth (6×10^{24} kg) and the mass of Jupiter (2×10^{27} kg) with all their zeroes on the board.
Deliberately squash up the zeroes in Jupiter to make the number look shorter.
Which planet do you think is "heavier"?

Answer: Pupils will probably still say Jupiter because they've seen pictures/diagrams that show it bigger – but bigger size *doesn't necessarily mean greater* mass.

Pupils will probably realise the attempted deception!

We need a better way of writing very big (and very small) numbers. You could write up the mass of an electron (9.1×10^{-31} kg) in decimal form – not easy to read or write like that.

The mass of the sun is 2×10^{30} *kg.*

1.15.3 People are used to decimal notation for large numbers; e.g., £2.3m for 2.3 million pounds. You can think of the 10^3 in 4.2×10^3, say, as either a multiplier (start with 4.2 and move all the digits 3 places to the left) or as a place value column heading (the 4 goes into the 10^3 column; i.e., the thousands column).

1.15.4 **NEED** *Guinness Book of Records* (or similar). Find numbers that you could write in standard form.

Could use this for a homework – pupils could use a newspaper or magazine or the internet if they don't have the book.

1.15.5 Work out how many seconds you've been alive, how many seconds till we go home at the end of term (do we finish early on the last day?), how many seconds of holiday till we come back, how many seconds till 12 noon on Christmas day or the new year, how many seconds till your birthday, how many seconds of maths lessons we have each week/fortnight, etc.

Answers will obviously vary:
For a 13-year-old, seconds alive is about $13 \times 365 \times 24 \times 60 \times 60 = 4.1 \times 10^8$.

It can be interesting to try to estimate the answers before working them out. We don't have a "feel" for size when we're in units that seem too small or too big.

1.15.6 Estimate how many water molecules you think there are in the swimming pool.

What do you need to know to work it out?
18 g of water (1 mole) contains Avogadro's number (6×10^{23}) molecules.
The density of water is 1 g/cm^3.

Answer:
Could consider the sloping bottom of the pool and treat the water as a trapezoidal prism, or just use average depth (e.g., 1.5 m);
e.g., take the pool as 50 m by 25 m by 1.5 m deep, so volume $= 50 \times 25 \times 1.5 = 1875$ m^3
$= 1.9 \times 10^9$ cm^3 $= 1.9 \times 10^9$ g, *so it contains*
$1.9 \times 10^9 \times 6 \times 10^{23} \div 18$ *molecules*
$= 6 \times 10^{31}$ *molecules (approx).*

Large and Small Numbers

prefix	symbol	size
tera-	T	10^{12}
giga-	G	10^{9}
mega-	M	10^{6}
kilo-	k	10^{3}
hecto-	h	10^{2}
deka-	D	10^{1}
	basic unit	10^{0}
deci-	d	10^{-1}
centi-	c	10^{-2}
milli-	m	10^{-3}
micro-	μ	10^{-6}
nano-	n	10^{-9}
pico-	p	10^{-12}
femto-	f	10^{-15}
atto-	a	10^{-18}

1 kilobyte is more than 1000 bytes – actually 1024 bytes because $2^{10} = 1024$

Distances – large and small	
10^{26} m	most distant galaxies
10^{21} m	size of our galaxy
10^{11} m	earth-sun distance
10^{9} m	sun (diameter)
10^{8} m	Jupiter (diameter)
10^{7} m	earth (diameter)
10^{4} m	black hole
10^{-4} m	smallest object visible to naked eye
10^{-6} m	smallest object visible under light microscope
10^{-8} m	large molecules
10^{-10} m	atoms
10^{-14} m	nucleus of atom
10^{-15} m	proton or neutron

10^{16} m = 1 light year (approx)
(speed of light = 3×10^{8} m/s, and a light year is the distance light travels in a vacuum in a year, so 1 light year
= $3 \times 10^{8} \times 60 \times 60 \times 24 \times 365$ metres)

US billion (10^{9}) used even by Bank of England nowadays; old UK billion (10^{12}) obsolete.

Names (and number of zeroes in brackets)

billion (9), trillion (12), quadrillion (15), quintillion (18), sextillion (21), septillion (24), octillion (27), nonillion (30), decillion (33), undecillion (36), etc.

googol (100); and a googolplex has a googol of zeroes (so it's $10^{\left(10^{100}\right)}$).
1 googol is more than the total number of protons, neutrons and electrons in the known universe!

milliard = billion (10^{9}) – useful when billion could mean 10^{9} or 10^{12}
lakh (10^{5}) and crore (10^{7}) are used in South Asia

1 million microphones = 1 phone
1 million phones = 1 megaphone
10 cards = 1 decacards
1 millionth of a fish = 1 microfish
10 millipedes = 1 centipede
10^{12} pins = 1 terrapin

1.16 Factors, Multiples, Prime Numbers and Divisibility

- **Factor** – an integer that goes into another integer exactly without any remainder.
  Need to be able to find them *all* for a particular integer – it's usually best to start at 1 and find them in pairs. Write them at opposite ends of the page so that you have them in order when you've finished.
  e.g., for 20 begin **1, 2, , 10, 20**
  You need only try every number up to the square root of the number n whose factors you're finding. Any number bigger than $\sqrt{n}$ would already have been found because its partner would be smaller than $\sqrt{n}$.
- **Multiple** – a number in the times table.
- **Prime Number** – a number with exactly *two factors* (1 and itself).
  By this definition 1 isn't prime because it hasn't got enough factors.
- Factors and multiples are opposites in the sense that if a is a factor of b then b is a multiple of a.
- Another difference is that all numbers have a finite number of factors but an infinite number of multiples.
- It can be useful to find the HCF and LCM by listing the factors/multiples before moving on to using prime factorisation to do it. It's helpful to find both the HCF and the LCM for the same pairs of numbers so that pupils are less likely to muddle up the two processes.
  HCF must be less than or equal to the smaller/smallest of the numbers; LCM must be more than or equal to the larger/largest of the numbers.

1.16.1 Find a square number that has fewer than 3 factors.
Find an even number that is prime.

Answers: 1 (only one factor –itself)

2 only

1.16.2 Shade in on a 100 square (see sheet) all the even numbers and the multiples of 3, 5 and 7 (not including 2, 3, 5 and 7 themselves).
The prime numbers are left.

There are infinitely many prime numbers. Euclid (about 330-270 BC) proved so by contradiction: Imagine there are only a certain number. Imagine multiplying them all together and adding 1. This new number either is prime (contradiction) or else it has a prime factor. But none of the prime numbers you've already got can be factors (all go in with remainder 1), so contradiction again. Therefore, proved.

See 100-square sheet.
There are 25 of them.
It's useful to have a list of prime numbers less than 100 by this means or some other (see sheet suitable for sticking into exercise books).

You only need to go up to multiples of 7 because 11 is the next prime number and $11^2 > 100$.
This process is sometimes called the "Sieve of Eratosthenes" (about 280 BC – 190 AD).

The logic of proof by contradiction can be appealing.

1.16.3 **NEED** scrap paper (A4).
Give each pupil (or pair) a sheet of scrap paper. Fold into 20 pieces (5 × 4, not as hard as it sounds!), tear up, write on the numbers 1-20.
Then, "Show me the multiples of 4", and pupils push those numbers into one spot.
"Show me the prime numbers greater than 7", etc.

This doesn't need to be done too carefully.

It's easy for the teacher to see how everyone is doing.

Can use this method for other number work; e.g., "Show me four numbers that add up to 15", or "four numbers with a mean of 8."

1.16.4 Draw a Venn diagram with subsets like "multiples of 4", "even numbers", "factors of 18", "prime numbers", etc.

Could extend to probability; e.g., "If you choose an integer at random from the set 1-20, what is the probability of choosing an even prime number?" Answer $\frac{1}{20}$.

1.16.5 Find a number with only 1 factor.
Find some numbers with exactly 2 factors.
Find some numbers with exactly 3 factors.
What's special about them?

Answers: only number 1
2, 3, 5, 7, etc. (prime numbers)
4, 9, 25, etc.
They're squares of prime numbers.
(If p is the prime number then the factors of p^2 are 1, p and p^2.)
Square numbers have an odd number of factors because they have a "repeated" factor.

What kind of numbers have n factors?
How can you decide how many factors a number will have without working them all out?

Suppose that p, q, r, ... are prime numbers, and a, b, c, ... are integers ≥ 0. Any power of a prime number can be written as p^a, and will have $a+1$ factors ($1, p, p^2, p^3, ...p^a$). Hence p has 2 factors and p^2 has 3 factors, as above.
Since every number can be factorised into primes, every number x can be written as $p^a q^b r^c ...$ and will have $(a+1)(b+1)(c+1)...$ factors, since any of the $a+1$ powers of p can be multiplied by any of the powers of the others.
So the possibilities are as below.

no. of factors	***prime factorisations of numbers that have that many factors***
1	1
2	p
3	p^2
4	p^3 or pq
5	p^4
6	p^5 or p^2q
7	p^6
8	p^7 or p^3q or pqr
9	p^8 or p^2q^2
10	p^9 or p^4q

You can set a challenge such as finding a number with exactly 13 factors.
One answer would be p^{12} (where p is a prime number), so choosing $p=3$ gives the number 531 441.
The factors are 1, 3, 9, 27, 81, 243, 729, 2 187, 6 561, 19 683, 59 049, 177 147 and 531 441, and there are thirteen of them.

So it's to do with the number of factors that n has (and n is how many factors the original number has)!

1.16.6 For prime factorisation, it's possible to draw tree diagrams going down the page.
Stop and put a ring around it when you reach a prime number.
Try each time to split the number into two factors that are as nearly equal as possible, because that leads to fewer steps.

e.g., for 24

```
        24
       /  \
      6    4
     / \  / \
    2   3 2  2
```

so 24 = $2^3 \times 3$

Could discuss why we don't count 1 as a prime number.

One reason is that prime factorisation would go on for ever! It's useful to have one unique way (apart from the order you write it in) of prime factorising every integer.

1.16.7 Tests for divisibility. Build up a big table of all the tests on the board (see sheet).
Then make 3 columns headed by 3 "random" numbers (e.g., 2016, 3465 and 2400) and put the numbers 1-12 down the side. Place either a tick or a cross in each column to say whether the column number is divisible by the row number.

It's productive to look for patterns in the answers (e.g., if a number is divisible by 10 then it's necessarily divisible by 2 and by 5, etc.).

Note for example that divisible by 4 $\Rightarrow$ divisible by 2, but not the other way round.

1.16.8 "I come into the classroom and ask the class to get into pairs – but 1 person is left over. We have to have groups of equal size, so I say never mind, instead get into groups of 3. Again 1 person is left over. We try groups of 4. Again, 1 person is left over. Finally groups of 5 works. How many people do you think there are in the class?"

Answer: 25 pupils; 85, 145, 205, etc. also work but hopefully would not be real class sizes!

One strategy is to say that $n-1$ must be a multiple of 2, 3 and 4; i.e., a multiple of 12, and go through the multiples of 12 until you find one where the number that is one higher is a multiple of 5.

1.16.9 "I'm thinking of a number …" or "I'm thinking of two numbers …"

Lots of possibilities.

1.16.10 Which number less than 100 has the most factors?

Answer: 96 (= $2^5 \times 3$), so 12 (= 6 × 2) factors: 1, 2, 3, 4, 6, 8, 12, 16, 24, 32, 48, 96.

Which has the fewest?

Answer: 1 (just has itself)

1.16.11 Perfect Numbers.
Classify each integer up to 20 as *perfect* (if it is the sum of all its factors apart from itself), *abundant* (if it is less than the sum of all its factors apart from itself) or *deficient* (if it is more). e.g., 6 = 1 + 2 + 3 so it's a perfect number, whereas the factors of 8 (apart from 8) add up to only 1 + 2 + 4 = 7, so it's deficient (but only just).

no.		**no.**		**no.**		**no.**	
1	d	**6**	p	**11**	d	**16**	d
2	d	**7**	d	**12**	a	**17**	d
3	d	**8**	d	**13**	d	**18**	a
4	d	**9**	d	**14**	d	**19**	d
5	d	**10**	d	**15**	d	**20**	a

Most numbers are deficient, because they don't have enough factors. Primes and powers of primes are all deficient, although powers of 2 are almost perfect (just too small by 1).
Perfect numbers are 6, 28, 496, 8218, etc.
Abundant numbers are 12, 18, 20, 24, 30, 36, etc.
Any multiple of a perfect or abundant number is abundant, and any factor of a perfect or deficient number is deficient.
No-one knows if there are any odd perfect numbers, but they've checked up to 10^{300} and haven't found any!

Why do you think we don't include the number itself when we add up the factors?
Because then you'd always get too much and it wouldn't be very interesting!

1.16.12 Which two numbers, neither containing any zeroes, multiply to make 100?
1000?
1 000 000?

Answers: 4 and 25

8 and 125
64 and 15625
In general, $10^x = 2^x \times 5^x$.

1.16.13 There are three brothers. The first one comes home 1 day in every 6 days, the second one once every 5 days and the third one once every 4 days. How often will they all be together?

Answer: The LCM of 6, 5 and 4 is 60.
So they'll meet every 60 days.

This assumes that the 1st and 3rd brothers don't perpetually miss each other (i.e., they start off all together). The numbers have to be pairwise co-prime to guarantee that they will all meet.

1.16.14 What is the smallest integer that all of the integers 1, 2, 3, 4, 5, 6, 7, 8 and 9 will go into?

Answer: The LCM of all the numbers 1 to 9 is $2^3 \times 3^2 \times 5 \times 7 = 2520$.

1.16.15 What is special about this number?
3816547290

Answer: Working from the left, the first digit is divisible by 1, the first two digits together (38) are divisible by 2, the first three digits are divisible by 3, etc. (And it uses the digits from 0 to 9 once each.)

Can you invent another number like it?

1.16.16 What collection of positive integers that add up to 100 (repeats allowed) makes the largest possible product when they are all multiplied together?

Answer: use two 2's and thirty-two 3's, and you get a total of 100 and a product of $2^2 \times 3^{32}$*, which is about 7 × 10*^{15}*.*
The logic behind this is that two 3's have a bigger product (9) than three 2's (8), so you would rather use 3's than 2's, but since 6's don't go into 100 you have to use two 2's to make the sum of 100.
Obviously you wouldn't use any 1's, since they wouldn't increase the product. There's no advantage using 4's, because 2 + 2 = 4 and 2 × 2 = 4. For any n *bigger than 4,* $2(n-2) > 4$*, so it's better to split up these numbers into two's.*

What about other totals? e.g., 16?

Answer: $3^4 \times 2^2 = 324$ *by the same logic.*

A related problem is to find the maximum product of *two numbers* (not necessarily integers) that sum to 100.
e.g., 10 + 90 = 100 and 10 × 90 = 900.
Can you split up 100 into two numbers that have a bigger product than this?

Answer: If the smaller of the two numbers is x*, then the product* p *will be* $p = x(100-x)$ *and this is a quadratic function with a maximum value of 2500 when* $x = 50$*.*
In general, if the total has to be t *then the maximum product is* $\frac{t}{2} \times \frac{t}{2} = \frac{t^2}{4}$*.*

What about multiplying together three or more numbers that sum to 100?

If n *numbers have a total of* t*, then their product will be a maximum when each of them is* $\frac{t}{n}$*, in which case their product will be* $\left(\frac{t}{n}\right)^n$*.*

1.16.17 Prove that all prime numbers > 3 are either one more or one less than a multiple of six.

This is a kind of proof by exhaustion. *We say that all integers can be written in one of 6 ways and we list them. Then we exclude types of number that we know could never be prime. All prime numbers must fit one of the two options left.*

Answer: Every number can be written as $6n$ *or* $6n+1$ *or* $6n+2$ *or* $6n+3$ *or* $6n+4$ *or* $6n+5$*. (*$6n+6$ *would be a multiple of 6, so could be written as just* $6n$*, etc.).*
Two's go into $6n$*,* $6n+2$ *and* $6n+4$ *(since both terms are multiples of 2), and threes go into* $6n$ *and* $6n+3$*. So the only numbers that* don't *have factors of 2 or 3 are* $6n+1$ *and* $6n+5$*. Not all of these will be prime, of course (e.g., 25 is* $6n+1$*), but all prime numbers (apart from 2 and 3) must take one of these forms.*

1.16.18 Make a chart to show factors of numbers up to, say, 20. Shade in the factors.
Look for patterns.

How will the pattern continue?

factor

no.	***1***	***2***	***3***	***4***	***5***	***6***	***7***	***8***	***9***	***10***
1	■									
2	■	■								
3	■		■							
4	■	■		■						
5	■				■					
6	■	■	■			■				
7	■						■			
8	■	■		■				■		
9	■		■						■	
10	■	■			■					■

1.16.19 Think of an integer between 1 and 9.
Add 1. Multiply by 9. Add up the digits.
Divide by 3. Add 1. What do you get?

Always 4. Why?

The digit sum of $9n$ *where* n *is an integer* ≤ 10 *is always 9. So when you divide by 3 and add 1 you will always get 4.*

1.16.20 Prisoners. A hundred prisoners are locked in their cells (numbered 1 to 100). The cell doors have special handles on the outside. When a guard turns the handle a locked door is unlocked and an unlocked door is locked.
All the doors start off locked. Then prison guard number 1 comes along and turns the handle of every door once (so all the doors are unlocked). Guard number 2 comes along and turns the handle of every *second* door (starting with cell 2). Guard 3 turns every third handle, and so on.
After the 100^{th} guard has been past, which prisoners can now get out of their cells?

Answer: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100
The answer is the square numbers since they have an odd number of factors, so their handles will have been turned an odd number of times.

Could try with a smaller number of cubes or pennies (heads up for "unlocked", tails for "locked").

1.16.21 Stamps. I have an unlimited number of 2p and 3p stamps. What possible (integer) values can I make?

Answers: With co-prime (HCF=1) values you can make all possible values beyond a certain number; e.g., with 2p and 3p, only 1 is impossible. Clearly 2p stamps enable all even amounts, and one 3p stamp plus any number of 2p stamps enable all odd amounts 3 or more. Hence everything except 1p.

What about with 3p and 4p stamps?

Harder now. Consider 3p stamps (since 3<4). If at any stage I can make 3 consecutive numbers, then from then on I can have any amount, by adding 3's to each. You can do 6 (= 3 + 3), 7 (= 3 + 4) and 8 (= 4 + 4), so the only impossibles are 1, 2 and 5.

What about 3p and 5p stamps?

Again, the first 3 consecutives you can make are 8 (3+5), 9 (3+3+3) and 10 (5+5), so the only impossibles are 1, 2, 4 and 7.

Variations on this include coins (obviously); fitting kitchen cabinets into different length kitchens; and scores with an unlimited number of darts on dartboards that have just a few different sections.

In general where x and y are co-prime, the highest impossible amount is $xy-(x+y)$.
(This is hard to prove, but see sheet.)

Functions like $A=3x+5y$ where x and y are integers are called Diophantine equations *(Diophantus, about 200-280 AD). A can be any integer if the co-efficients (3 and 5) are co-prime (HCF=1), but x and y may need to be negative.*

This is a very rich investigation which can run and run. It can turn into a Fibonacci-type problem by asking how many ways there are of making, say, 50p from 5p and 7p stamps. You can let the order matter by saying that you are buying the stamps at a post office and the assistant is annoyingly pushing the stamps under the safety screen one at a time.
So 5p + 7p and 7p + 5p count as different ways of making 12p.
A spreadsheet might help.

Answer: 57 ways.
The key thing to notice is that if n_c is the number of ways of making a total of c pence out of x pence and y pence stamps, then assuming that $c>x$ and $c>y$, $n_c=n_{c-x}+n_{c-y}$. (because you add one x pence or one y pence stamp). For example, if $x=5$ and $y=7$ then $n_{17}=3$ and $n_{19}=3$ so $n_{24}=6$.

You could make a tree diagram starting …

```
        50
      /    \
    43      45
   /  \    /  \
  36   38 38   40
```

… to count up the number of ways.

(See the table of answers on the sheet – highest impossible amounts are shaded in.)

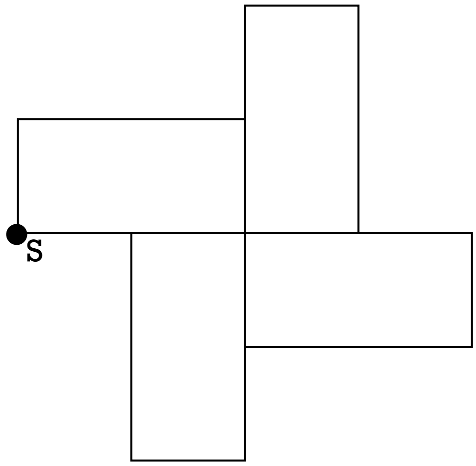
1.16.22 Spirolaterals.
Make a Vedic Square by writing the numbers 1 to 9 around the edge of an 8 × 8 square, multiplying and filling in the boxes.
Then re-draw it finding the *digit sums* of the answers.
This gives

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
2	4	6	8	1	3	5	7	9
3	6	9	3	6	9	3	6	9
4	8	3	7	2	6	1	5	9
5	1	6	2	7	3	8	4	9
6	3	9	6	3	9	6	3	9
7	5	3	1	8	6	4	2	9
8	7	6	5	4	3	2	1	9
9	9	9	9	9	9	9	9	9

Then begin somewhere in the middle of a sheet of A4 0.5 cm × 0.5 cm squared paper. Mark a dot. Choose a column in the table, say the 3's. Draw a line 3 squares long straight up the paper, turn right, draw a line 6 squares long, turn right, then 9, then 3, and so on down the column, always turning *right* (see drawing on the right – S is the starting point). When you get to the bottom of a column of numbers, just start again at the top.
Keep going until you get back to the place where you started.

The easiest columns to do are 3, 6, 9 (but 9 is boring – a square, obviously).

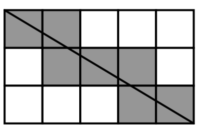
Islamic art is based on Vedic squares.
(Digit sum = *sum of the digits of the numbers.)*



All of them will fit on A4 0.5 cm × 0.5 cm squared paper, but you need to start in a sensible place. Teacher may need to assist.

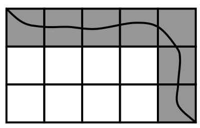
The tricky thing is seeing that turning right when coming down *the page towards you means actually going left. Some people find it helpful to turn the paper as you go (like turning a map), so you're always drawing away from you, and some people just find that confusing. It's also useful to mention the name "spirolateral" so we expect it to "spiral in", not meander further and further towards the edge of the paper!*

1.16.23 Diagonals in Rectangles.
If an $x \times y$ rectangle is drawn on squared paper, how many squares does the diagonal line pass through?
x and y are integers.
e.g., for a 5×3 rectangle, the line goes through 7 squares.



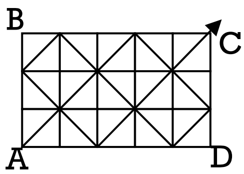
We count it as "going through" a square even if it only *just* clips the square. Only if it goes *exactly* through a crossing-point do we not count the square.

Answer:
If x and y are co-prime (HCF=1) then the line must go through $x+y-1$ squares. This is the smallest number of squares between opposite corners. (Imagine a curly line – see below).



If $HCF(x,y)>1$ *then every* $\frac{1}{HCF(x,y)}$ *of the way along the diagonal line will be a crossing point, since* $\frac{x}{HCF(x,y)}$ *and* $\frac{y}{HCF(x,y)}$ *will be integers.*
Each of these $HCF(x,y)-1$ *crossing points means one fewer square for the diagonal to go through. So that means the total number of squares will be*
$x+y-1-(HCF(x,y)-1)$
$=x+y-HCF(x,y)$

1.16.24 Snooker investigation.
A snooker ball is projected from the near left corner (A, below) of a rectangular snooker table at 45° to the sides. If there are pockets at all four corners, and the table has dimensions $x \times y$ (x is the horizontal width), which pocket will the ball end up in?
x and y are integers.



Assume that every time the ball hits a side it rebounds at 45°, and that the ball never runs out of kinetic energy.

Above for a 5×3 table, the answer is pocket C.

Try some examples on squared paper.

Hint:
How many diagonal steps will the ball move before it lands in a pocket?

Answer:
Every diagonal step forward moves the ball 1 square horizontally and 1 square vertically. Since x is the horizontal distance, after $x, 3x, 5x, \ldots$ "steps" the ball will be at the right wall. After $2x, 4x, 6x, \ldots$ steps the ball will be at the left wall.
Similarly, after $y, 3y, 5y, \ldots$ steps the ball will be at the top side. After $2y, 4y, 6y, \ldots$ steps the ball will be at the bottom side.
Therefore, the first time that a multiple of x is equal to a multiple of y (i.e., after $LCM(x, y)$ steps), the ball will be in one of the corners.

It will never be corner A, because the ball only reaches there if it travels an even number of x's and an even number of y's. That will never be the LCM of x and y because half that many x's would match half that many y's.

If $\frac{LCM(x, y)}{x}$ is odd, then the pocket will be either C or D. Otherwise, it will be A or B.
If $\frac{LCM(x, y)}{y}$ is odd, then the pocket will be either B or C. Otherwise, it will be A or D.
Taken together, this means you can always predict which pocket the ball will end up in.

e.g., for a 10×4 table, $LCM(10, 4) = 20$
$\frac{20}{10}$ is even so AB side; $\frac{20}{4}$ is odd so BC side.
Hence pocket B.

1.16.25 What is the significance of the digit sum of an integer when the integer isn't a multiple of 9? Try working some out.

Find out what "casting out 9's" refers to. (Suitable for a homework: ask grandparents.)

You can prove that the digit sum of a 3-digit number, say, is the remainder when dividing by 9 by writing "abc", as $n = 100a + 10b + c$. (This process would work just as well however many digits the number had.)

$$\begin{aligned} n &= 100a + 10b + c \\ &= 99a + 9b + a + b + c \\ &= 9(11a + b) + (a + b + c) \end{aligned}$$

so $a + b + c$ is the remainder when n is divided by 9. (This assumes that $a + b + c < 9$. If it's *equal* to 9, then n is a multiple of 9; if it's *more than* 9, then we can just start again and find the digit sum of *this* number, because the remainder of this number when divided by 9 will be the same as the remainder of n when divided by 9.)

Answer: It's the remainder when you divide the number by 9. E.g., $382 \div 9 = 42$, remainder 4. And the digit sum of 382 is 4 (actually 13, but the digit sum of 13 is 4).
(The digit sum of a multiple of 9 is itself a multiple of 9.)

Hence the method of casting out 9's" in which every integer in the calculation is replaced by its digit sum. When the calculation is done with these numbers, the answer should be the digit sum of the answer to the original question. This provides a way of checking.
e.g., $946 + 326 = 1272$, replacing 946 and 326 by their digit sums gives $1 + 2 = 3$, and 3 is the digit sum of 1272. This doesn't guarantee that the sum is correct, but if this test doesn't *work then the sum is* definitely *wrong.*

This is a bit like the modern "check-sums" method used on bar-codes to make sure the machine has read the numbers accurately. Here a single mistake can always be identified.

1.16.26 Why not define "highest common multiple" and "lowest common factor", as well?

Answer: If you think about it, LCF would always be 1 and HCM would always be infinite!

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Prime numbers less than 100

2	3	5	7	11
13	17	19	23	29
31	37	41	43	47
53	59	61	67	71
73	79	83	89	97

Prime numbers less than 100

2	3	5	7	11
13	17	19	23	29
31	37	41	43	47
53	59	61	67	71
73	79	83	89	97

Prime numbers less than 100

2	3	5	7	11
13	17	19	23	29
31	37	41	43	47
53	59	61	67	71
73	79	83	89	97

Prime numbers less than 100

2	3	5	7	11
13	17	19	23	29
31	37	41	43	47
53	59	61	67	71
73	79	83	89	97

Prime numbers less than 100

2	3	5	7	11
13	17	19	23	29
31	37	41	43	47
53	59	61	67	71
73	79	83	89	97

Prime numbers less than 100

2	3	5	7	11
13	17	19	23	29
31	37	41	43	47
53	59	61	67	71
73	79	83	89	97

Prime numbers less than 100

2	3	5	7	11
13	17	19	23	29
31	37	41	43	47
53	59	61	67	71
73	79	83	89	97

Prime numbers less than 100

2	3	5	7	11
13	17	19	23	29
31	37	41	43	47
53	59	61	67	71
73	79	83	89	97

Prime numbers less than 100

2	3	5	7	11
13	17	19	23	29
31	37	41	43	47
53	59	61	67	71
73	79	83	89	97

Prime numbers less than 100

2	3	5	7	11
13	17	19	23	29
31	37	41	43	47
53	59	61	67	71
73	79	83	89	97

Prime Numbers

2	3	5	7	11	13	17	19	23	29
31	37	41	43	47	53	59	61	67	71
73	79	83	89	97	101	103	107	109	113
127	131	137	139	149	151	157	163	167	173
179	181	191	193	197	199	211	223	227	229
233	239	241	251	257	263	269	271	277	281
283	293	307	311	313	317	331	337	347	349
353	359	367	373	379	383	389	397	401	409
419	421	431	433	439	443	449	457	461	463
467	479	487	491	499	503	509	521	523	541
547	557	563	569	571	577	587	593	599	601
607	613	617	619	631	641	643	647	653	659
661	673	677	683	691	701	709	719	727	733
739	743	751	757	761	769	773	787	797	809
811	821	823	827	829	839	853	857	859	863
877	881	883	887	907	911	919	929	937	941
947	953	967	971	977	983	991	997	1009	1013
1019	1021	1031	1033	1039	1049	1051	1061	1063	1069
1087	1091	1093	1097	1103	1109	1117	1123	1129	1151
1153	1163	1171	1181	1187	1193	1201	1213	1217	1223
1229	1231	1237	1249	1259	1277	1279	1283	1289	1291
1297	1301	1303	1307	1319	1321	1327	1361	1367	1373
1381	1399	1409	1423	1427	1429	1433	1439	1447	1451
1453	1459	1471	1481	1483	1487	1489	1493	1499	1511
1523	1531	1543	1549	1553	1559	1567	1571	1579	1583
1597	1601	1607	1609	1613	1619	1621	1627	1637	1657
1663	1667	1669	1693	1697	1699	1709	1721	1723	1733
1741	1747	1753	1759	1777	1783	1787	1789	1801	1811
1823	1831	1847	1861	1867	1871	1873	1877	1879	1889
1901	1907	1913	1931	1933	1949	1951	1973	1979	1987
1993	1997	1999	2003	2011	2017	2027	2029	2039	2053
2063	2069	2081	2083	2087	2089	2099	2111	2113	2129
2131	2137	2141	2143	2153	2161	2179	2203	2207	2213
2221	2237	2239	2243	2251	2267	2269	2273	2281	2287
2293	2297	2309	2311	2333	2339	2341	2347	2351	2357
2371	2377	2381	2383	2389	2393	2399	2411	2417	2423
2437	2441	2447	2459	2467	2473	2477	2503	2521	2531
2539	2543	2549	2551	2557	2579	2591	2593	2609	2617
2621	2633	2647	2657	2659	2663	2671	2677	2683	2687
2689	2693	2699	2707	2711	2713	2719	2729	2731	2741
2749	2753	2767	2777	2789	2791	2797	2801	2803	2819
2833	2837	2843	2851	2857	2861	2879	2887	2897	2903
2909	2917	2927	2939	2953	2957	2963	2969	2971	2999
3001	3011	3019	3023	3037	3041	3049	3061	3067	3079
3083	3089	3109	3119	3121	3137	3163	3167	3169	3181
3187	3191	3203	3209	3217	3221	3229	3251	3253	3257
3259	3271	3299	3301	3307	3313	3319	3323	3329	3331
3343	3347	3359	3361	3371	3373	3389	3391	3407	3413
3433	3449	3457	3461	3463	3467	3469	3491	3499	3511
3517	3527	3529	3533	3539	3541	3547	3557	3559	3571
3581	3583	3593	3607	3613	3617	3623	3631	3637	3643
3659	3671	3673	3677	3691	3697	3701	3709	3719	3727
3733	3739	3761	3767	3769	3779	3793	3797	3803	3821
3823	3833	3847	3851	3853	3863	3877	3881	3889	3907
3911	3917	3919	3923	3929	3931	3943	3947	3967	3989
4001	4003	4007	4013	4019	4021	4027	4049	4051	4057

Tests for Divisibility

integer	test	divides into the number if...
1	no test	any integer
2	look at units digit	0, 2, 4, 6 or 8
3	digit sum →→	3, 6, 9
4	look at last 2 digits	divisible by 4
5	look at units digit	0 or 5
6	test for 2 and test for 3	passes both tests
7	double the units digit and subtract from the rest of the number →→	divisible by 7
8	divide it by 2	divisible by 4
9	digit sum →→	9
10	look at units digit	0
11	alternating digit sum (…+–+–+)→→	0

Notes

- →→ means do the same thing to the answer and keep going until you have only 1 digit left
- "digit sum →→" on 732 gives 732 → 12 → 3 (so passes the test for 3)
- "alternating digit sum (…+–+–+)→→" on 698786 gives + 6 – 8 + 7 – 8 + 9 – 6 (working from *right to left* and beginning with +) = 0 (so passes the test for 11)
- "double the units digit and subtract from the rest of the number →→" on 39396 gives 3939 – 12 = 3927 → 392 – 14 = 378 → 37 – 16 = 21, divisible by 7 (so passes the test for 7)

- You can use a calculator to generate multiples to use for practice.
  (E.g., type in a 4-digit "random" integer, multiply by 7 and you have a "random" multiple of 7 for trying out the test for divisibility by 7.)
- A test for divisibility by 12 could be to pass the tests for divisibility by 3 and by 4.
  It wouldn't be any good to use the tests for divisibility by 2 and by 6 because passing the test for 6 means that the number must be even so the test for 2 adds nothing.
  3 and 4 are co-prime (HCF = 1), but 6 and 2 aren't.

These tests are worth memorising and practising.

Stamps Investigation (Proof)

If the stamps are x pence and y pence (x and y co-prime), then the highest impossible value (with unlimited quantities of each) is $xy - x - y$ pence.

This is easy to show for a particular pair of stamp values.

For example, for $x = 3$ and $y = 5$ we write down all the positive integers using three columns (see right – imagine the columns going down for ever).

1	2	3
4	5	6
7	8	9
10	11	12
13	14	15
16	17	18
…	…	…

We're going to shade in all the numbers that are *possible*.
Clearly the right column will all be possible by using just the 3p stamps, because the numbers are all multiples of 3.

In the second row, 5 will be possible (one 5p stamp) so we shade that in. Also everything under 5 in the second column will be possible by using one 5p stamp and different numbers of 3p stamps.

Similarly, 10 (two 5p stamps) and everything below it will be possible.
So 7 is the highest impossible amount (and this is $3 \times 5 - 3 - 5$).

Now we try the same thing generally, with x pence and y pence stamps (x and y co-prime).

The diagram is shown on the right.
We will assume that $x < y$.
As before, the right column will all be possible (multiples of x pence), so we shade it in.

1	2	3	…	$x-1$	x
$x+1$	$x+2$	$x+3$	…	$2x-1$	$2x$
$2x+1$	$2x+2$	$2x+3$	…	$3x-1$	$3x$
…	…	…	…	…	…

The column containing y will all be possible from y downwards (y, $y+x$, $y+2x$, …) and this cannot be the same column as the x column (x, $2x$, $3x$, …) because y is not a multiple of x (they're co-prime).

For the same reason, $2y$ cannot be in either of the two columns dealt with so far (unless there are only two columns because $x = 2$). (It can't be in the "x column" because $2y$ cannot be a multiple of x, and the extra y places moved on from y can't equal a multiple of x, which would be necessary if it were in the same column as y).
So we keep locating the next multiple of y, and we always find it in a previously unvisited column, and we shade it in and we shade in the rest of that column below it.

There are x columns altogether, so when we reach $(x-1)y$ and shade that in (and all the numbers beneath it) the highest impossible amount will be the number directly above $(x-1)y$ (since all the other numbers in that row will already be shaded in).
This number will be $(x-1)y - x = xy - x - y$, and so this will be the highest impossible value.

Number of ways n_c of making up a cost c pence out of stamps

c	**5p, 7p**	**3p, 7p**	**5p, 8p**	**5p, 9p**	**7p, 10p**
1	0	0	0	0	0
2	0	0	0	0	0
3	0	1	0	0	0
4	0	0	0	0	0
5	1	0	1	1	0
6	0	1	0	0	0
7	1	1	0	0	1
8	0	0	1	0	0
9	0	1	0	1	0
10	1	2	1	1	1
11	0	0	0	0	0
12	2	1	0	0	0
13	0	3	2	0	0
14	1	1	0	2	1
15	1	1	1	1	0
16	0	4	1	0	0
17	3	3	0	0	2
18	0	1	3	1	0
19	3	5	0	3	0
20	1	6	1	1	1
21	1	2	3	0	1
22	4	6	0	0	0
23	0	10	4	3	0
24	6	5	1	4	3
25	1	7	1	1	0
26	4	15	6	0	0
27	5	11	0	1	3
28	1	9	5	6	1
29	10	21	4	5	0
30	1	21	1	1	1
31	10	14	10	0	4
32	6	28	1	4	0
33	5	36	6	10	0
34	15	25	10	6	6
35	2	37	1	1	1
36	20	57	15	1	0
37	7	46	5	10	4
38	15	51	7	15	5
39	21	85	20	7	0
40	7	82	2	1	1
41	35	76	21	5	10
42	9	122	15	20	1
43	35	139	8	21	0
44	28	122	35	8	10
45	22	173	7	2	6
46	56	224	28	15	0
47	16	204	35	35	5
48	70	249	10	28	15
49	37	346	56	9	1
50	57	343	22	7	1

continued ...					
51	84	371	36	35	20
52	38	519	70	56	7
53	126	567	17	36	0
54	53	575	84	11	15
55	127	768	57	22	21
56	121	913	46	70	1
57	95	918	126	84	6
58	210	1139	39	45	35
59	91	1432	120	18	8
60	253	1485	127	57	1
61	174	1714	63	126	35
62	222	2200	210	120	28
63	331	2398	96	56	1
64	186	2632	166	40	21
65	463	3339	253	127	56
66	265	3830	102	210	9
67	475	4117	330	165	7
68	505	5053	223	74	70
69	408	6030	229	97	36
70	794	6515	463	253	2
71	451	7685	198	330	56
72	938	9369	496	221	84
73	770	10345	476	114	10
74	883	11802	331	224	28
75	1299	14422	793	463	126
76	859	16375	421	495	45
77	1732	18317	725	295	9
78	1221	22107	939	211	126
79	1821	25744	529	477	120
80	2069	28662	1289	793	12
81	1742	33909	897	716	84
82	3031	40166	1056	409	210
83	2080	45037	1732	435	55
84	3553	52226	950	940	37
85	3290	62273	2014	1288	252
86	3563	70781	1836	1011	165
87	5100	80888	1585	620	21
88	3822	96182	3021	912	210
89	6584	110947	1847	1733	330
90	5370	125925	3070	2004	67
91	7116	148408	3568	1420	121
92	8390	173220	2535	1055	462
93	7385	196706	5035	1852	220
94	11684	229296	3683	3021	58
95	9192	269402	4655	3015	462
96	13700	307653	6589	2040	495
97	13760	355221	4382	1967	88
98	14501	417810	8105	3585	331
99	20074	480873	7251	5025	792
100	16577	551927	7190	4435	287

1.17 Triangle Numbers

- The n^{th} triangle number is $\frac{1}{2}n(n+1)$. The first few are 1, 3, 6, 10, 15, 21, …
  The difference between each number and the next goes up by 1 each time.
  The formula $\frac{1}{2}n(n-1)$ gives the $(n-1)^{\text{th}}$ triangle number for $n \geq 2$.
  The n^{th} triangle number is the same as ${}^{n+1}C_2$.
- Triangle numbers can be illustrated with equilateral or right-angled isosceles triangles of dots.
  We include 1 because it completes the pattern of differences and fits the above formulas.
  For the same reason, we say that 1 is the first *square number*, since 1 × 1 = 1.

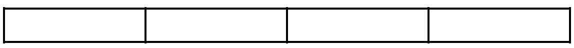
1.17.1 Handshakes.
If everyone in the room were to shake hands with everyone else in the room, how many handshakes would there be?
Start small. Try 3 people. Stand up and do it.
Now try a group of 4. Somebody keep count.

If there are n people, each of them needs to shake hands with everyone except themselves; i.e., $n-1$ people, so that makes $n(n-1)$, but this counts every handshake twice "from both ends". So the answer is $\frac{1}{2}n(n-1)$.

An alternative context is sending "Christmas cards" (or similar) – if everyone sends one to everyone else how many cards will there be? This avoids the problem of double-counting, since if A sends a card to B, B (out of politeness!) will send a card to A.
You can do this with teams playing matches if you say that every team wants to play every other team "*at home*".

Each person has to send $n-1$ cards, because they send one to everybody except themselves. There are n people that do that, so $n(n-1)$ cards altogether.
If n teams all play each other "at home", then all n venues are visited by all $n-1$ other teams (obviously they don't play against themselves!) and that is just $n(n-1)$ games.

1.17.2 Rectangles. How many rectangles (of any size) are there in this shape?
There are more than 4.



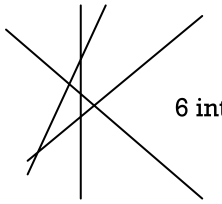
Answer: 10, the 4th triangle number

You can explain it by thinking about what happens when you add another block on to the right end to make 5 blocks. The 5th block makes 5 more rectangles: itself, itself and the one to its left, itself and the two to its left, itself and the three to it's left, and itself and the four to its left. This happened because there were already 4 rectangles there.
In general, adding the n^{th} rectangle increases the total by n.

Can extend to 2 dimensions (i.e., like a chessboard). Here, a square counts as a rectangle.

An array of rectangles $x \times y$ will contain a total of $\frac{1}{2}x(x+1) \times \frac{1}{2}y(y+1)$ rectangles $= \frac{1}{4}xy(x+1)(y+1)$. This is because the first row contains $\frac{1}{2}x(x+1)$ different rectangles, and each of these has height 1, so $\frac{1}{2}y(y+1)$ of them can be fitted vertically down the grid.
If $x = y$ (a square array, like a chessboard), this reduces to $= \frac{1}{4}x^2(x+1)^2$, so when $x = 8$ (for a chessboard), the total is 1296 rectangles (the sum of the 1st eight cube numbers).
(See section 1.14.11.)

1.17.3 Straight lines and intersections. How many crossing-points are there when 4 lines overlap if each new line is drawn so that it crosses as many lines as possible?



Need to use A4 paper at least and choose the angles of the lines wisely.
It's easy to miscount or draw lines which don't cross the maximum possible number of lines.

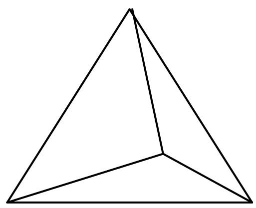
The n^{th} line should cross $n-1$ lines, so after the n^{th} line there should be $\frac{1}{2}n(n-1)$ intersections. (Every line crosses $n-1$ others, making $n(n-1)$ crossings, but this double-counts every line, so we put in a factor of $\frac{1}{2}$.)
Or you can say that there will be a crossing-point for each pair of lines, so it's the number of ways of choosing 2 from n, or ${}^nC_2 = \frac{n(n-1)}{2!}$, which is the same.

How many regions are produced at each stage? (Count 1 line as producing 2 regions.)
A context for this is cutting up a cake:
"What is the maximum number of pieces you can divide a cake into using 5 straight cuts? The pieces don't have to be equal sizes."
Start small – try 1, 2, 3 cuts first.

regions = pieces of cake after n cuts = $\frac{1}{2}(n^2+n+2) = \frac{1}{2}n(n+1)+1$; i.e., one more than the n^{th} triangle number.

See a similar investigation (about the maximum number of triangles *formed from intersecting lines) in section 1.19.12.*

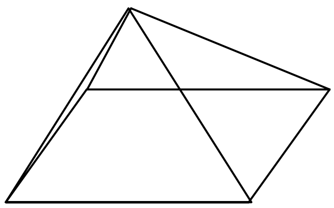
1.17.4 Baked Bean Tins.
These can be stacked in a triangular array, giving the triangle numbers. This is easy to extend to *tetrahedron numbers* by making a triangular horizontal layer and putting a smaller triangular layer on top until you reach one can at the very top. The finished stack is roughly tetrahedral.



The n^{th} ***tetrahedron number*** *is the sum of the first n triangle numbers, so they go 1, 4, 10, 20, 35, 56, 84, …; in general the n^{th} one is $\frac{1}{6}n(n+1)(n+2)$.*

Why does this formula always give a positive integer whenever n is a positive integer?
Answer: Because n, $n+1$ and $n+2$ are consecutive integers. This means that either one of them or two of them must be even, so when you multiply all three you're bound to get a number that is divisible by 2. Also, one of them will always be a multiple of 3 (because multiples of 3 come every three numbers), so the product will be divisible by 3. So when you divide by 6 you get an integer.

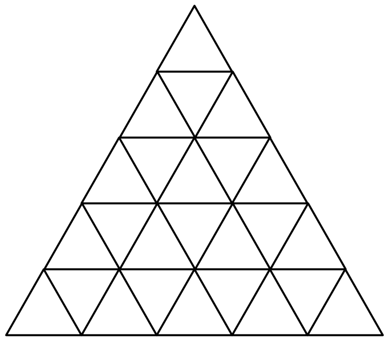
Making each layer a *square* array of cans instead, gives the *square pyramidal numbers*.



The n^{th} ***square pyramidal number*** *is the sum of the first n square numbers, so they go 1, 5, 14, 30, 55, 91, 140, …; in general the n^{th} one is $\frac{1}{6}n(n+1)(2n+1)$.*

Why does this formula give an integer value?
Answer: Either n or $n+1$ must be even, so the product of the three numbers must be divisible by 2. If either of these numbers is a multiple of 3, then of course it will be fine. If neither is, then n must be of the form $3m+1$, where m is an integer, and that means $n+1=3m+2$ and $2n+1=2(3m+1)+1=6m+3$, which is divisible by 3. So one of the numbers will always be divisible by 3, and one divisible by 2, so dividing by 6 must give an integer answer.

1.17.5 How many *small* triangles are there altogether in this drawing?



Answer: 25.
Start with fewer rows and look for a pattern. Can count triangles with a side at the bottom (type 1) and triangles with a point at the bottom (type 2) separately (shade in one set).

no. of rows of triangles	*no. of triangles (type 1)*	*no. of triangles (type 2)*	*total no. of triangles*
1	1	0	1
2	3	1	4
3	6	3	9
4	10	6	16
5	15	10	25
6	21	15	36
7	28	21	49

so if there are n rows of small triangles, the total number of triangles = n^2.
So we see that the sum of two consecutive triangle numbers is a square number, or

$$\frac{n(n-1)}{2}+\frac{(n+1)n}{2}=\frac{n(n-1+n+1)}{2}$$

$$=\frac{2n^2}{2}=n^2$$

or $T_{n-1}+T_n=S_n$.

Another way to count them is row-by-row. This shows that the sum of the first n odd numbers is the n^{th} square number.

Because the large equilateral triangle is mathematically similar to the small ones, as the number of rows increases as n the area (the number of small equilateral triangles) increases as n^2.

How many triangles (of any size) are there altogether in the drawing?

(This is much harder.)

Answer: 48.
Again, start with a drawing with fewer rows and look for a pattern.

no. of rows of triangles	*no. of triangles (type 1)*	*no. of triangles (type 2)*	*total no. of triangles*
1	1	0	1
2	4	1	5
3	10	3	13
4	20	7	27
5	35	13	48
6	56	22	78
7	84	34	118

After n rows the number of type 1 triangles is the sum of the first n triangle numbers.

The sum of the 1st n triangle numbers is the n^{th} tetrahedral number $\frac{1}{6}n(n+1)(n+2)$ (see section 1.17.4).

The number of type 2 triangles is more complicated, because you only get a bigger type 2 triangle on every other *row (a bigger type 1 triangle appears with* every *new row). So the number of type 2 triangles depends on whether n is even (then it's the sum of the alternate triangle numbers starting with the first one (1)) or odd (then it's the sum of the alternate triangle numbers but beginning with the second one (3)).*

For n even, the number of type 2 triangles is $\frac{1}{24}n(n+2)(2n-1)$.
For n odd, the number of type 2 triangles is $\frac{1}{24}(n+1)(2n^2+n-3)$.

Adding $\frac{1}{6}n(n+1)(n+2)$ to each of these gives the formulas for the total number of triangles when n is even and when n is odd.

So the total number of triangles is $\frac{1}{8}n(n+2)(2n+1)$ if n is even, and $\frac{1}{8}(n+1)(2n^2+3n-1)$ if n is odd.

1.17.6 Which integers from 1 to 20 can be made from the sum of just 2 triangle numbers?
(The triangle numbers themselves can obviously be made from just one triangle number.)

Answer:
All except 5, 8, 14, 17 and 19.

no.	sum	no.	sum	no.	sum
1	= 1	2	= 1 + 1	3	= 3
4	= 3 + 1	5	= 3 + 1 + 1	6	= 6
7	= 6 + 1	8	= 6 + 1 + 1	9	= 6 + 3
10	= 10	11	= 10 + 1	12	= 6 + 6
13	= 10 + 3	14	= 10 + 3 + 1	15	= 15
16	= 15 + 1	17	= 15 + 1 + 1	18	= 15 + 3
19	= 15 + 3 + 1	20	= 10 + 10		

and other possibilities.

Gauss (1777-1855) proved that every *integer is the sum of at most three triangle numbers.*

Which numbers can be expressed as the sum of two *consecutive* triangle numbers?

4, 9, 16; i.e., the square numbers (see section 1.17.5).

1.17.7 Polygon Numbers.
Can you define pentagon, hexagon, heptagon, etc. numbers?
Find different formulas for the different n^{th} polygon numbers.

Answer:
Make the length of each side increase by 1 as n goes up by 1.
If p is the number of sides of the polygon (so $p=3$ for the triangle numbers), then the n^{th} p-gon number is $\frac{1}{2}n(n(p-2)-(p-4))$, and for $n=1$ this reduces to 1 no matter what the p value. So 1 is the first number in all the polygon number sequences.

This can also be written as $\frac{1}{2}n(np-2n-p+4)$.

What kind of numbers are "rectangle numbers"?
If you put $p=2$ into the formula you just get n.

Rectangle numbers would be either all the integers (if you allow rectangles with a side of length 1), or all non-prime numbers (if you don't). (Non-primes are called "composites".)

1.17.8 Mystic Rose.
Space four points evenly around the circumference of a circle. Join every point to every other point.
How many lines do you need?

Answer: 6 (four sides of the square plus two diagonals)
For n points, the number of lines is the $(n-1)^{th}$ triangle number, $\frac{1}{2}n(n-1)$. This happens because each of the n points is connected to the other $n-1$ points, making $n(n-1)$ lines, but this double-counts each line (from both ends) so we put in the factor of $\frac{1}{2}$.

With more points it makes beautiful patterns that are suitable for display work.

Finding the number of regions is deceptively difficult – see section 1.19.11.

1.17.9 When we use the formula $\frac{1}{2}n(n+1)$ (where n is an integer), why do we always get an integer answer?
(Picking just any two integers and multiplying them and dividing by 2 won't necessarily give an integer.)

n and $n+1$ are consecutive *numbers, and that means that one is always odd and the other even. Therefore you're always multiplying an odd number by an even number, and that always gives an even answer (multiplying any integer by an even number always gives an even answer). So when you halve the answer you get an integer.*
Another way to think about it is that $\frac{1}{2}n(n+1)=\frac{n}{2}(n+1)=n\frac{(n+1)}{2}$; i.e., you can halve either n or $n+1$ (whichever is even) and multiply the answer by the other one, so you'll always get an integer.

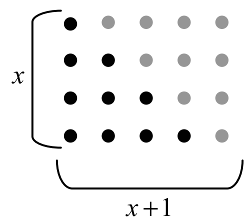
1.17.10 Prove that 8 times any triangle number is 1 less than a square number.

Answer: $8\times\frac{1}{2}n(n+1)=4n(n+1)=4n^2+4n$, *which is 1 less than* $(2n+1)^2$, *so it's always 1 less than an odd square.*

1.17.11 Consecutive Sums.
What is the total of all the integers from 1 to 100? What about 1 to 1000? Or more?

You can tell the story of Gauss (1777-1855) who was told by a teacher to add up all the numbers from 1 to 100 (without a calculator, of course, in those days) and did it very quickly.

Pupils may think that it ought to be possible to replace repeated addition by some sort of multiplication, and that is what we're doing.



Answers: $S_{1-10}=55$, $S_{1-100}=5050$ *and* $S_{1-1000}=500500$, *etc.*
(where S_{a-b} *means the sum of all the integers from* a *to* b *inclusive; i.e.,* $\sum_{i=a}^{b} i$ *).*

Various approaches:

1. *Realise by drawing dots or squares, that* S_{1-x} *is the* x^{th} *triangle number. Use the formula* $\frac{1}{2}x(x+1)$ *or see this by combining two identical triangles of dots and getting a rectangle* x *by* $(x+1)$.
2. *Say that the "average" value (actually the mean and median) of the numbers from 1 to* x *must be* $\frac{1+x}{2}$, *and since there are* x *values the total must be* $\frac{x(x+1)}{2}$.
3. *Pair up 1 and* x *(to make* $x+1$*), 2 and* $x-1$ *(to make* $x+1$ *also), etc. Eventually you have* $\frac{1}{2}x$ *pairings (if* x *is even), so the total is* $\frac{x(x+1)}{2}$. *(But what if* x *is odd? Then there are* $\frac{x-1}{2}$ *pairings and the middle number (* $\frac{x+1}{2}$ *) is left over. So the total this time is* $\frac{x-1}{2}\times(x+1)+\frac{x+1}{2}=\frac{x+1}{2}(x-1+1)=\frac{x(x+1)}{2}$, *the same.)*
4. *Find* $S_{1-10}=55$ *by some method (or just add them up) and argue that* S_{11-20} *must be 100 more, because each number in the sum is 10 more than each number in the first sum (writing out some of it makes this clearer). So altogether* S_{11-20} *must be* 10×10 *bigger* $=155$. *Now* $S_{1-20}=55+155=210$, *so* $S_{21-40}=210+20\times20=610$ *(by similar reasoning), and so* $S_{1-40}=820$, *and so on, giving* $S_{41-80}=2420$, $S_{1-80}=3240$, $S_{81-100}=1810$ $(=S_{1-20}+20\times80)$, *so* $S_{1-100}=3240+1810=5050$. *Although this method is not quick, it does involve some good thinking.*

Which integers is it possible to make using the sum of consecutive positive integers?

See sheet "Sums of Consecutive Integers". Clearly all odd numbers (except 1) are possible (see diagonal line in the table), since pairs of consecutive integers added together make all of the odd numbers.
Impossible totals are 1, 2, 4, 8, 16, … (powers of 2). You can see this because in the formula $T_b-T_a=\frac{1}{2}(b-a)(a+b+1)$, *one bracket must be even and one odd, so* T_b-T_a *has at least one odd factor, and so can't be a power of 2.*

Sums of Consecutive Integers

The sums of consecutive integers are the *trapezium numbers*, which are the differences between (non-consecutive) triangle numbers.

This happens because $\sum_{x=1}^{a} x$ is the a^{th} triangle number (T_a) and $\sum_{x=1}^{b} x$ is the b^{th} triangle number (T_b). $[\,T_n = \frac{1}{2}n(n+1)\,]$

So $\sum_{x=a+1}^{b} x$ (the sum of the consecutive integers from $a+1$ to b) is $T_b - T_a$, where $a \geq 0$ and $b - a > 1$. $[\,T_b - T_a = \frac{1}{2}(b-a)(a+b+1)\,]$

The possible integer sums are given in *italics* in the table below (horizontal triangle number minus vertical triangle number).

		T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}	T_{11}	T_{12}	T_{13}	T_{14}	T_{15}	T_{16}	T_{17}	T_{18}	T_{19}	T_{20}
		1	**3**	**6**	**10**	**15**	**21**	**28**	**36**	**45**	**55**	**66**	**78**	**91**	**105**	**120**	**136**	**153**	**171**	**190**	**210**
T_1	**1**			5	9	14	20	27	35	44	54	65	77	90	104	119	135	152	170	189	209
T_2	**3**				7	12	18	25	33	42	52	63	75	88	102	117	133	150	168	187	207
T_3	**6**					9	15	22	30	39	49	60	72	85	99	114	130	147	165	184	204
T_4	**10**						11	18	26	35	45	56	68	81	95	110	126	143	161	180	200
T_5	**15**							13	21	30	40	51	63	76	90	105	121	138	156	175	195
T_6	**21**								15	24	34	45	57	70	84	99	115	132	150	169	189
T_7	**28**									17	27	38	50	63	77	92	108	125	143	162	182
T_8	**36**										19	30	42	55	69	84	100	117	135	154	174
T_9	**45**											21	33	46	60	75	91	108	126	145	165
T_{10}	**55**												23	36	50	65	81	98	116	135	155
T_{11}	**66**													25	39	54	70	87	105	124	144
T_{12}	**78**														27	42	58	75	93	112	132
T_{13}	**91**															29	45	62	80	99	119
T_{14}	**105**																31	48	66	85	105
T_{15}	**120**																	33	51	70	90
T_{16}	**136**																		35	54	74
T_{17}	**153**																			37	57
T_{18}	**171**																				39

1.18 Trial and Improvement

- Whether a problem is sensible to solve by trial and improvement depends on what other methods the pupils have met. Using trial and improvement to find numbers like $\sqrt{20}$, for example (as some textbooks do), is pointless for pupils who know about the square root button. Similarly, solving quadratics by trial and improvement will be unmotivating for pupils who know about the formula.
  In general it's better to choose problems that are difficult to solve by other methods. The only difficulty then is that the teacher has no easy access to the answers! It's possible to use graphical calculators or spreadsheets to find the solutions, and on the following page are some accurate solutions to various cubic and quadratic equations.

1.18.1 One way to begin is by plotting a graph of $y = x^2 + x$ for $0 \le x \le 5$ and $0 \le y \le 30$.

Answer: 2.70156212…

This task is suitable for pupils who don't know a method for solving quadratic equations. If the pupils have already covered this, use a cubic equation instead.

x	0	1	2	3	4	5
y	0	2	6	12	20	30

If we know x , working out y is easy.
But going the other way is hard.
If $y = 10$, what is x ?
It's hard to work out (try it by algebra), but easy to check if we think we know the answer.

Lots of things are like this: public key encryption works on the principle that it's easy to encode something but very hard to decode it unless you know the key.
Differentiation is usually much easier than integration.

Pupils can read off an approximate value from the graph.
Could draw a more accurate graph from, say, $x = 2.5$ to $x = 3$ to get a better approximation.

But we could work it out without bothering to draw the graph. We'd need to assume that there's a continuous curve throughout the region.

x	2		3
y	6		12

too small too big

Using a horizontal table such as this is more visually helpful for some pupils than a normal table of rows. It has similarity to a number-line.

Use this to find x to 2 dp when $y = 22$.

Answer: 4.21699057… or 4.22 (2 dp) – not 4.21

Can you find a negative solution, also to 2 dp?

Answer: -5.21699057… or -5.22 (2 dp)

1.18.2 Some thought needs to be given to how to get an answer to a particular degree of accuracy; say, 3 dp.

For example if the answer is between, say, 4.216 and 4.217, we just need to try 4.2165; if 4.2165 is too small then the answer is 4.217 to 3 dp. Otherwise the answer is 4.216 to 3 dp.
This works because if 4.2165 is too small then whatever the answer is it is between 4.2165 and 4.217, and all those numbers round to 4.217 to 3 dp.

1.18.3 Can you invent a graph where this method *wouldn't* find an answer?
(Here the graph doesn't go as low as $y = -2$.)

A discontinuous curve will cause problems. Otherwise the only problem will be cases where there is no solution; e.g., $x^2 + x = -2$.

Solutions to Cubic Equations

The method of Cardano (1545) for solving $x^3 + mx = n$ gives

$$x = \sqrt[3]{\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}} - \sqrt[3]{-\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}}$$

This leads to the values in the table below. These can be used to check trial and improvement work. Always request a particular degree of accuracy (e.g., 3 dp) for answers.

	equation	x
1	$x^3 + x = 1$	0.682327804
2	$x^3 + 2x = 1$	0.453397652
3	$x^3 + 3x = 1$	0.322185355
4	$x^3 + 4x = 1$	0.246266172
5	$x^3 + x = 2$	1.000000000
6	$x^3 + x = 3$	1.213411663
7	$x^3 + x = 4$	1.378796700
8	$x^3 + 2x = 2$	0.770916997
9	$x^3 + 2x = 3$	1.000000000
10	$x^3 + 2x = 4$	1.179509025

	equation	x
11	$x^3 - x = -1$	–1.324717957
12	$x^3 - x = 1$	1.324717957
13	$x^3 - x = 2$	1.521379707
14	$x^3 - x = 3$	1.671699882
15	$x^3 + 10x + 5 = 0$	–0.488353313
16	$x^3 = 5x + 10$	2.905474006
17	$x^3 + 12x = 10$	0.791942868
18	$x^3 + 5x = 7$	1.119437527
19	$x^3 = 11 + x$	2.373649822
20	$x^3 + 6x - 9 = 0$	1.206959814

Solutions to Quadratic Equations

The solutions below come from using the equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

so there are two solutions (x_A and x_B) for each equation.

The problems can be set as equations in x to solve or they can be converted into words; e.g., the equation $x(x+3) = 30$ is equivalent to the question "the product of two numbers is 30 and the difference between the numbers is 3, what are the two numbers?"; e.g., areas of rectangles given the difference between the lengths of the sides. Answers should be given to a particular degree of accuracy (e.g., 3 dp).
Depending on the context, negative answers may or may not be acceptable.

	equations		x_A	x_B
1	$x(x+3) = 30$	$x^2 + 3x - 30 = 0$	$x =$ 4.178908346 $x + 3 =$ 7.178908346	$x =$ –7.178908346 $x + 3 =$ –4.178908346
2	$x(x+10) = 50$	$x^2 + 10x - 50 = 0$	$x =$ 3.660254038 $x + 10 =$ 13.66025404	$x =$ –13.66025404 $x + 10 =$ –3.660254038
3	$x(20-x) = 50$	$-x^2 + 20x - 50 = 0$	$x =$ 2.928932188 $20 - x =$ 17.07106781	$x =$ 2.928932188 $20 - x =$ 17.07106781
4	$x(x+2) = 25$	$x^2 + 2x - 25 = 0$	$x =$ 4.099019514 $x + 2 =$ 6.099019514	$x =$ –6.099019514 $x + 2 =$ –4.099019514
5	$x(x+1) = 10$	$x^2 + x - 10 = 0$	$x =$ 2.701562119 $x + 1 =$ 3.701562119	$x =$ –3.701562119 $x + 1 =$ –2.701562119
6	$x(2x+1) = 12$	$2x^2 + x - 12 = 0$	$x =$ 2.21221445 $2x + 1 =$ 5.424428901	$x =$ –2.71221445 $2x + 1 =$ –4.424428901

1.19 Sequences

- With some pupils it may be worth avoiding statements like "now we're going to do algebra" because of the reputation algebra has for being "hard"! You can say later, "you've been doing algebra!"
- It's necessary to distinguish between *term-to-term* rules plus starting value(s) (i.e., *inductive* definitions like $u_n = n+3$; $u_1 = 1$) and *position-to-term* rules (i.e., *deductive* definitions like $u_n = 3n-2$, the same sequence). Neither is necessarily "better"; they're just different ways of describing sequences.
- Pupils sometimes assume that if a rule works for the first few terms then it will always work. (This is probably because we often use such straightforward sequences.) "Points and Regions" (section 1.19.11) is a helpful investigation to do to show that this isn't always the case.
- *Proving* a rule needs some insight into what is going on. A clear diagram often helps, or looking at the problem from another angle. Comments of this kind appear on the right below.
  Pupils often confuse *proving* a result for *all values* of n with *checking* a result for a *few particular values* of n.

1.19.1 "James, give me a number between 1 and 10."
Whatever the pupil says, the teacher doubles and adds 1 and writes it on the board.
Eventually ask, "What's going on?"
Can use quite difficult functions.
e.g., $y = 3x-1$, $y = 10-x$, $y = 42+x$

Teacher as "function machine".

Another system is for the teacher to have a red board pen and the volunteer pupil a black pen. The pupil records the pupils' numbers in black in the left column of the table; the teacher records the teacher's number in red in the right column of the table.
After a while we decide that it would be better to choose black numbers in order, so next time instead of choosing the black numbers randomly we just write 1-5 in order (and the pupil can sit down!).

black	*red*
4	*17*
1	*2*
7	*32*

Using black and red colours to represent sets of numbers is helpful.
You can eventually write, e.g.,
black = 5 × red – 3 and then b = 5 × r – 3 and then $b = 5r-3$.

Pupils can invent their own table of numbers or you can present one on the board:

black	red	yellow	green	purple	white
1	11	5	16	3	8
2	12	10	22	8	11
3	13	15	28	13	14
4	14	20	34	18	17
5	15	25	40	23	20

It's worth keeping the colours going for a bit, but soon you can start saying things like "I don't have a white pen, so I'll write 'white' in black" and pupils will soon accept the "colour" even if it's all written in black.
(This work is slightly easier on a blackboard than a whiteboard because you generally have more different colours available.)

Avoid embarrassing a pupil who is colour-blind.

Look for connections – always horizontal, not down the columns. "I want a statement about how one colour is connected to another colour", "How do you get from the black numbers to the red numbers?", etc.

e.g.,
(or write using function machines or in words)
$r = b+10$ *(not* $b+1$*)*; $y = 5\times b$; $g = r+y$;
$p = y-2$; $w = \frac{g}{2}$; *etc.*
Can you get from b to w?
× 3 then + 5; could write using function machines or perhaps $w = 3b+5$.

Pupils can make up their own tables of numbers with rules for getting from one colour to another.
(See "Making Formulas" sheet.)

These formulas can then be "formalised" into $w = 3y+2$, *etc.*

1.19.2 "Matchstick problems", or similar.
Readily available in books. Pupils have to find a formula to give the number of lines (or matchsticks) used in a sequence of drawings.

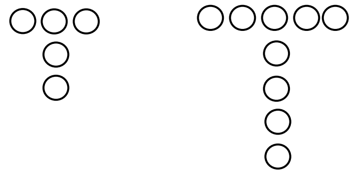
Need to establish that n stands for the "pattern number", so $n=1$ for the first drawing, and so on, and that u_n (or whatever symbol) represents the n^{th} term; i.e., how many matchsticks are used in the n^{th} pattern.

1.19.3 **NEED** linking cubes. Cube Animals.
Make a simple dog/horse/giraffe, etc. from about 8 linking cubes. Make a second "larger" one of the same "species" but using more cubes. What would the third one look like? How many cubes would it need for its legs?, etc.
What would the 100th one look like? If you had 100 cubes and you wanted to make the biggest model you could, how many would you use for the head?

You can use different colours for different parts of the animal (e.g., red for the head, blue for the body, green for the legs, yellow for the tail) and work out formulas for each colour; e.g., if n is the model number then the number of cubes in the head (red) could be $2n$, perhaps. If you add the expressions for the number of cubes in the different body parts you get the formula for the total number of cubes needed for the whole animal.

1.19.4 Investigate letters of the alphabet made out of squares or circles; e.g., AEFHILNTVY.
Different pupils may use different rules about how to get the next letter (some may be enlarging, and others stretching), but that needn't matter.

e.g., letter T



1.19.5 A general method for finding the n^{th} term involves looking for common differences.
The argument here is that if the numbers go up in 3's, say, then the sequence must have something to do with the 3 times table (because the numbers in the 3 times table go up in 3's).
There are 3 different sequences of integers that go up in 3's: the 3 × table; the 3 × table shifted on 1; and the 3 × table shifted on 2. (The 3 times table shifted on 3 is the same as the 3 × table except that the first term is lost.)
It's sensible to call these $3n$, $3n+1$ and $3n+2$ (or $3n-1$).
(See "Finding the Formula" sheet.)

Could split the class into three groups and "chant" each of these sequences separately; then put them together to see that it makes 1, 2, 3, 4, 5, 6, 7, 8, …

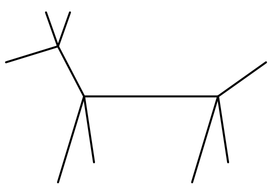
"Finding the Formula" answers:

1a	$t=7n$	***1b***	$t=5n-1$
2a	$t=3n-2$	***2b***	$t=n+14$
3a	$t=10n+5$	***3b***	$t=2n+22$
4a	$t=5n+8$	***4b***	$t=n-1$
5a	$t=4n+308$	***5b***	$t=3n-2.5$
6a	$t=4n+997$	***6b***	$t=1000n+1$
7a	$t=6n+1$	***7b***	$t=2$
8a	$t=5-n$ *or* $t=-n+5$	***8b***	$t=28-2n$

1.19.6 There are lots of interesting investigations to do at this stage; e.g., "Loops" (see sheet, although sheet certainly not necessary).

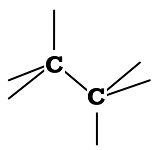
Answers to "Loops":
c increases by 1 each time,
and $a=2c$ and $b=c+1$.

A similar one is "Stick-Animals".
Like "stick-people", these are made from short straight lines joined together at their ends. No loops are allowed.
e.g., stick-dogs:



Count the number of lines (10), the number of junctions (3) and the number of "ends" (8).
What is the connection between these numbers?

Drawings must be clear.



Some similarity with chemical molecular structures (aliphatic hydrocarbons – compounds containing only carbon and hydrogen atoms and no rings – acyclic); e.g., ethane is C_2H_6; the two carbon atoms are like junctions and the six hydrogens like ends, and the number of chemical bonds is 2+6–1=7.

junctions + ends = lines + 1

1.19.7 Quadratic, Cubic and Beyond.
A linear sequence: $u_n = an + b$

$$\begin{array}{ccccc} a+b & & 2a+b & & 3a+b \\ & a & & a & \end{array}$$

A quadratic sequence: $u_n = an^2 + bn + c$

$$\begin{array}{ccccc} a+b+c & & 4a+2b+c & & 9a+3b+c \\ & 3a+b & & 5a+b & \\ & & 2a & & \end{array}$$

A similar method will work for a cubic sequence $u_n = an^3 + bn^2 + cn + d$ (or higher polynomial sequences).

An alternative is to write, say for a cubic,

$$u_n = a + b(n-1) + c(n-1)(n-2) + d(n-1)(n-2)(n-3).$$

If the first *differences are constant, then the sequence is* linear, *and* $u_n = an + b$ *(see left). You can work out* a *and* b *because* a *is the common difference and the first term is* $a+b$*, or you can find the "zeroth" term and that is* b*.*

If the first differences aren't *constant, you find the* second *differences. If they're constant, then the sequence is quadratic, and* $u_n = an^2 + bn + c$ *(see left). You can work out* a*,* b *and* c *because the common (second) difference is* $2a$ *(so you can work out* a*). You can use this value of* a *together with the first of the first differences (*$3a+b$*) to find* b *and then use this with the first term (*$a+b+c$*) to find* c*.*

You substitute $n=1$ *to find* a*,* $n=2$ *to find* b *and so on, but then you have to expand the brackets and simplify to get your final formula.*

1.19.8 Ever-Increasing Rectangles.
Start with 3 squares in a line (shaded at the centre of the diagram on the right) and surround them with other squares (just 8 more, touching each side but not at the corners – the white ones on the right).
Keep going, surrounding the shape you have at each stage with squares on the edges.
Record the number of squares in each layer.

(It's a very good idea to shade in alternate layers so that they don't get muddled up.)

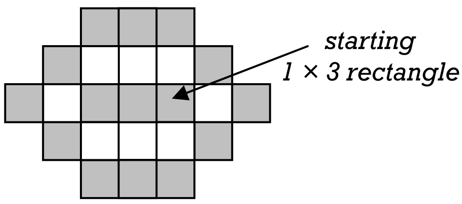
Investigate different shaped rectangles.
How many squares would be in the n^{th} layer surrounding an x by y rectangle?

The total number of squares after n *layers is given by* $2n^2 - 6n + 4 + 2(n-1)(x+y) + xy$*.*

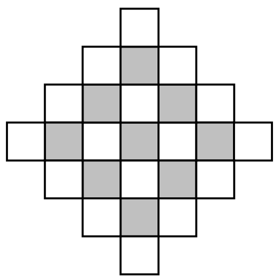
If you start with a 1 × 1 square ($x = y = 1$*), then a simpler pattern emerges (see right). If you look at the diagram at 45°, you can see that the total number of squares is the sum of two square numbers (3 × 3 = 9 grey squares and 4 × 4 = 16 white squares).*
In general, the total number of squares after n *layers is* $n^2 + (n-1)^2 = 2n^2 - 2n + 1$*.*
(The first square counts as layer 1.)

Can extend by trying hexagons instead (on isometric paper). Or you can use other non-rectangular shapes made of squares.
(It works best if the hexagons you begin with join side-to-side, not corner-to-corner, so if you want to use the paper in landscape orientation, then you need to have the dots the other way round from normal – hence the "unusual" dotty paper included – see sheet.)

Can make nice display work.



For a 1 × 3 rectangle, the n^{th} *layer has* $4n+4$ *squares.*
In general, for an x *by* y *rectangle, the* n^{th} *layer will have* $4(n-1) + 2(x+y)$ *squares.*
A drawing makes it clear that the first term comes from the "corners", and the second from the "sides".



For a row of x *hexagons, the* n^{th} *layer contains* $6n + 2(x-1)$ *hexagons, and the total number of squares after* n *layers is* $(3n-2)(n-1) + (2n-1)x$*.*

Or you can extend to cubes *in 3-dimensions. For* n *3-dimensional "shells", the total number of cubes is* $\frac{1}{3}(4n^3 - 6n^2 + 8n - 3)$*.*

1.19.9 Dots in Rectangles.
Draw a 3 × 4 rectangle on some 0.5 cm × 0.5 cm squared paper. (You could use square dotty paper, but it isn't necessary)
How many dots are there inside it? (Count a dot as any place where the grid lines cross).

There are 6 dots, and it's easy to generalise that an x by y rectangle will contain $(x-1)(y-1)$ dots.

What about tilted rectangles?
Try ones that are at 45° to start with.
Call them "gradient 1" because the sides with positive gradient have a gradient of 1.

The other sides have a gradient of –1. In general if one pair of sides have gradient m, the other pair will have gradient $-\frac{1}{m}$, but that's another investigation!

Try rectangles of width 2 (this means 2 diagonal spaces) and find a formula for the number of dots inside as the length of the rectangle changes.

For "gradient 1" and $y=2$, $n=3x-1$.

Find an overall formula for an x by y "gradient 1" rectangle.

For "gradient 1" rectangles, $n=2xy-x-y+1$.

Extend to rectangles of gradient 2, 3 and m.

For "gradient 2", $n=5xy-x-y+1$.
For general "gradient m" rectangles, $n=(1+m^2)xy-x-y+1$.
Notice how all these formulas are symmetrical in x and y (if you swap around x and y you get the same formula).

1.19.10 Try the differences method on the Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21, …
(Each term is the sum of the two previous terms.)

Every set of differences are just the Fibonacci sequence again. Although the term-to-term rule is simple ($u_n=u_{n-1}+u_{n-2}$), the n^{th} term is given by

$$u_n=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right],$$

and this is fairly hard to prove.
Fibonacci (1170-1250) wrote a famous book called Liber Abaci.

Which other sequences will behave like this when you try to find common differences?

Answer: doubling sequences like 1, 2, 4, 8, 16, 32, 64, … or 5, 10, 20, 40, 80, …

1.19.11 Points and Regions.
If 10 points are spaced evenly around the circumference of a circle, and every point is joined to every other point with a straight line, how many pieces is the circle divided into?
Start with a small number of points and do BIG drawings (A4 paper at least). Put a cross in those areas you've counted so you don't get muddled up. (If you need to re-count the same drawing, use a different colour for the crosses.) Work out a formula for the number of regions you get if you begin with n points.

This is a good task to use to emphasise the need not to make assumptions too early on!

no. of points	***no. of regions***
1	*1*
2	*2*
3	*4*
4	*8*
5	*16*
6	***31***
7	***57***

This is about as far as you can get on an A4 sheet of paper counting carefully, but it's enough data.

Note 31 and not 32. The pattern ***isn't*** 2^{n-1}.

4^{th} differences turn out to be constant (1), so the equation is quartic.

The solution is that for n points the number of regions = $\frac{1}{24}(n^4-6n^3+23n^2-18n+24)$.

Answer for 10 points is 256. (Ironically this is a power of 2; actually 2^8.)

The reason why is that the total number of regions must be 1 + number of chords + number of intersections inside the circle. There are ${}^nC_2$ chords because each chord has two ends and ${}^nC_4$ intersections because each is defined by 4 points on the circle. This gives the formula $1+{}^nC_2+{}^nC_4$, which is the same.

For further details, see the sheet, which illustrates a different way of obtaining a formula from a number sequence. Although more work, this method shows how each term can be worked out separately.

Pupils could invent a sequence that looks as though it follows as simple pattern but deviates later; e.g., $\frac{1}{6}(n^3-6n^2+29n-6)$ goes 3, 6, 9, **13, 19, 28,** …

1.19.12 Triangles.
(See related investigation in section 1.17.3.)

If n straight lines intersect, what is the maximum possible number of triangles created?

Answer: any 3 lines could make a triangle, so it will be the number of ways of choosing 3 lines out of n; in other words,

$${}^nC_3 = \frac{n(n-1)(n-2)}{3!} = \tfrac{1}{6}(n^3 - 3n^2 + 2n).$$

1.19.13 Frogs. (A well-known investigation.)
Have 4 boys and 3 girls sitting on chairs in a row with an empty chair in between.

B B B B _ G G G

They are the "frogs". They have to swap around so that the final arrangement is

G G G _ B B B B

The only two types of move allowed are

- a *slide* of 1 place (only) into an adjacent empty seat; and
- a *jump* over 1 person (only) into an empty seat.

The question is can it be done, and if so what is the minimum possible number of moves. What if the numbers of boys and girls change?

Pupils can use counters or cubes (blue for boys, green for girls) and record the moves on squared paper.

You can record results in a 2-way table:

number of girls g \ number of boys b	1	2	3	4
1	3	5	7	9
2	5	8	11	14
3	7	11	15	19
4	9	14	19	24

(There are other sequences of moves when $b = 1$ that are also a minimum.)

Notice that the table is symmetrical about the diagonal, and the formulas are symmetrical so that if you swap b for g you get the same formula.

Note that this doesn't prove that such a minimum sequence of moves (i.e., with nobody wasting a move by going backwards) will be possible; only that if it is this is how many moves it will take.

Getting the minimum number of moves isn't always easy. Recording the pattern of slides and jumps helps to see what to do next. Pupils could state their conclusions about how to get the fewest moves; e.g.,

- *The pattern should always have some symmetry; e.g., for 4 boys and 3 girls it is SJSJJSJJJSJJJSJJSJS (19 moves).*
  (S = slide, J = jump)
- *No boy or girl ever needs to go "backwards".*
- *Try starting by sliding what you have most of and then jumping and sliding (once each) the other.*
- *Always jump after a slide, and jump as much as possible (because a jump moves each person twice as far as a slide but still only counts as one move)!*

Let b = number of boys, g = number of girls and n = minimum number of moves.
In general,

$$n = (b+1)(g+1) - 1$$
$$= bg + b + g$$

So when $b = g$, $n = (b+1)^2 - 1$; i.e., 1 less than the next square number.

If $b = 1$, then it's fairly easy to see that $n = 2g + 1$. First the boy slides, then the 1st girl jumps, then the boy slides again and the 2nd girl jumps and so on until the gth girl jumps ($2g$ moves by then – each girl has jumped once and the boy has slid g times). Finally the boy has to slide once to get to the end seat. So altogether $2g + 1$ moves (g slides and $g + 1$ jumps).

When $b > 1$ and $g > 1$, it's harder to justify the number of moves necessary. It's possible to argue like this: There must be a total of bg jumps (each boy has to jump over or be jumped over by each girl). From the start, each girl has to move $b + 1$ places to get to her final position, and each boy has to move $g + 1$ places (the +1 because of the empty seat they all have to pass). So the minimum total number of shifts must be $g(b+1) + b(g+1) = 2bg + b + g$. But the bg jumps will make up $2bg$ shifts (each jump moves a frog 2 places), so the number of slides must be the remaining $b + g$. So the total number of moves must be $bg + b + g$ of which bg are the jumps and $b + g$ are the slides.

1.19.14 Rectangles in Grids (see sheet).

Very rich investigation.

1.19.15 Stopping Distances (see sheet).
(The figures have been adjusted slightly to fit a quadratic formula.)

Answers:

1. *We imagine that the driver has a fixed (average) "thinking time", regardless of the car's speed, so if travelling twice as fast, he'll cover twice the distance in that thinking time.*
2. *20 mph = 32 kph = 9 m/s, so thinking time = $6 \div 9$ = about 0.7 seconds. This would probably represent quite an alert driver.*
3. *When the speed doubles, the braking distance goes up by a factor of about 4. This happens because kinetic energy ($\frac{1}{2}mv^2$) is proportional to the square of the speed v (m is the mass), and assuming that the brakes apply a constant slowing force they will need 4 times as much distance to reduce the kinetic energy to zero.*

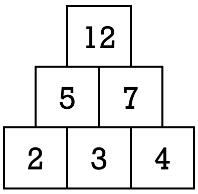
Answers (continued):

4. *If v is the speed (mph), then in metres, thinking distance = $\frac{3v}{10} = 0.3v$;*
   braking distance = $2\left(\frac{v}{10}\right)^2 - \frac{2v}{5} + 8$
   $= 0.02v^2 - 0.4v + 8$; and
   overall stopping distance = $2\left(\frac{v}{10}\right)^2 - \frac{v}{10} + 8$
   $= 0.02v^2 - 0.1v + 8$.
5. *At 80 mph, overall distance = 128 metres.*
6. *The thinking distance would be about the same, but the braking distance could be ten times as much, or even more.*
7. *The condition of the car (tyres, brakes, mass, etc.) and the driver (tired, alcohol, distractions, experience, skill, etc.).*

1.19.16 Pyramids. (A well-known investigation.)

Pyramid shapes are made out of a triangle number of rectangular boxes.
Each rectangle contains a number equal to the sum of the numbers in the two rectangles underneath it.
The numbers in the bottom row are consecutive integers.
Work out a way to predict the number in the rectangle on the top (the peak number) given the numbers at the bottom for different heights of pyramids.

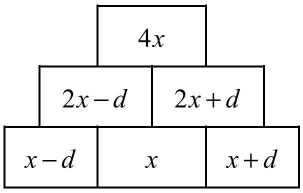
e.g., How can you predict 12 given (2,3,4) without working out the other numbers?



Hint: Start with *odd* numbers of rows.

(See sheet of blank pyramids that may save time.)

Answer: 12 = 4 × 3.
In general, for a 3-row pyramid, the peak number = $4x$, where x is the middle number on the bottom row.



This works even if the bottom row aren't consecutive numbers, so long as they go up with a common difference (d above).

To deal with larger pyramids (with odd numbers of rows), treat them as containing this three-row unit. The middle number of a row goes up 4 times when you jump up 2 rows.
For n (odd) rows, peak number = $2^{n-1}m$.

Even-height pyramids (n even) need considering separately. There is no middle number at the bottom this time.
In general, if x is the smallest number *in the bottom row (so x is the number at the* end *now, not in the middle), and d is the common difference along the bottom row, then the peak number = $\frac{1}{4}2^n(2x + d(n-1))$. In fact this works whether n is odd or even so long as x is the smallest number in the bottom row.*

1.19.17 **NEED** Graphical Calculators.
Use the ANS feature to generate different sequences; e.g., 5, 8, 11, 14, …

Linear sequences are easy, but you can get the square numbers, for example, if you start with 1 and use the inductive formula $u_n = (\sqrt{u_{n-1}} + 1)^2$.

Making Formulas

Here are some numbers. Each column of numbers is called by the name of a colour.

black	red	yellow	green	purple	white
1	3	2	4	6	8
2	6	3	7	9	11
3	9	4	10	12	14
4	12	5	13	15	17
5	15	6	16	18	20

There is a connection between the black numbers and the red numbers.
If you multiply a black number by 3 you get the corresponding red number.
We can write this as a formula or using number machines.
Look at line 1 in the list below.
Complete the list.

	Formulas	**Number Machines**
1	red = 3 × black	black $\rightarrow \boxed{\times 3} \rightarrow$ red
2	yellow =	black $\rightarrow \boxed{+ 1} \rightarrow$ yellow
3	green =	red $\rightarrow \boxed{\quad} \rightarrow$ green
4	green =	black $\rightarrow \boxed{\times \quad} \rightarrow \boxed{+ \quad} \rightarrow$ green
5	purple =	yellow $\rightarrow \boxed{\quad} \rightarrow$ purple
6	purple =	red $\rightarrow \boxed{\quad} \rightarrow$ purple
7	purple =	green $\rightarrow \boxed{\quad} \rightarrow$ purple
8	purple =	black $\rightarrow \boxed{\quad} \rightarrow \boxed{\quad} \rightarrow$ purple
9	white =	purple $\rightarrow \boxed{\quad} \rightarrow$ white
10	white =	green $\rightarrow \boxed{\quad} \rightarrow$ white
11	white =	yellow $\rightarrow \boxed{\quad} \rightarrow \boxed{\quad} \rightarrow$ white
12	white =	red $\rightarrow \boxed{\quad} \rightarrow$ white

Extra Tasks

- Write some more formulas connecting these colours.
- Try to write a formula that starts **black =**

Making Formulas **ANSWERS**

Here are some numbers. Each column of numbers is called by the name of a colour.

black	red	yellow	green	purple	white
1	3	2	4	6	8
2	6	3	7	9	11
3	9	4	10	12	14
4	12	5	13	15	17
5	15	6	16	18	20

There is a connection between the black numbers and the red numbers.
If you multiply a black number by 3 you get the corresponding red number.
We can write this as a formula or using number machines.
Look at line 1 in the list below.
Complete the list.

	Formulas	**Number Machines**
1	red = **3 × black**	black → $\boxed{\times 3}$ → red
2	yellow = $b + 1$	black → $\boxed{+ 1}$ → yellow
3	green = $r + 1$	red → $\boxed{+ 1}$ → green
4	green = $3b + 1$	black → $\boxed{\times 3}$ → $\boxed{+ 1}$ → green
5	purple = $3y$	yellow → $\boxed{\times 3}$ → purple
6	purple = $r + 3$	red → $\boxed{+ 3}$ → purple
7	purple = $g + 2$	green → $\boxed{+ 2}$ → purple
8	purple = $3b + 3$	black → $\boxed{\times 3}$ → $\boxed{+ 3}$ → purple
9	white = $p + 2$	purple → $\boxed{+ 2}$ → white
10	white = $g + 4$	green → $\boxed{+ 4}$ → white
11	white = $3y + 2$	yellow → $\boxed{\times 3}$ → $\boxed{+ 2}$ → white
12	white = $r + 5$	red → $\boxed{+ 5}$ → white

Extra Tasks

- Write some more formulas connecting these colours.
- Write a formula that starts **black =** for example, **black = yellow – 1**

Finding the Formula

Work out formulas that fit the numbers in these tables.
Check that your formulas work by trying them when $n = 3$.

1

a

n	1	2	3	4
t	7	14	21	28

b

n	1	2	3	4
t	4	9	14	19

2

a

n	1	2	3	4
t	1	4	7	10

b

n	1	2	3	4
t	15	16	17	18

3

a

n	1	2	3	4
t	15	25	35	45

b

n	1	2	3	4
t	24	26	28	30

4

a

n	1	2	3	4
t	13	18	23	28

b

n	1	2	3	4
t	0	1	2	3

5

a

n	1	2	3	4
t	312	316	320	324

b

n	1	2	3	4
t	0.5	3.5	6.5	9.5

6

a

n	1	2	3	4
t	1001	1005	1009	1013

b

n	1	2	3	4
t	1001	2001	3001	4001

7

a

n	1	2	3	4
t	7	13	19	25

b

n	1	2	3	4
t	2	2	2	2

8

a

n	1	2	3	4
t	4	3	2	1

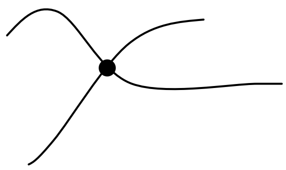
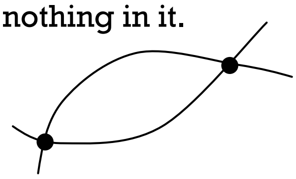
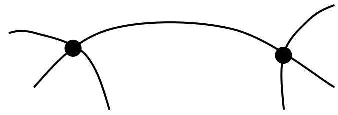
b

n	1	2	3	4
t	26	24	22	20

Extra Task Make up a table of numbers like this that fit a more tricky formula.

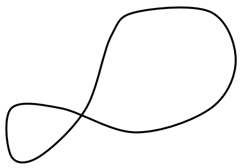
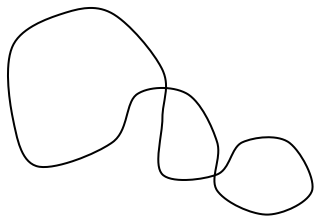
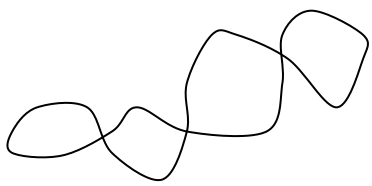
Loops

This task is about **crossovers**, **blobs** and **arcs**.

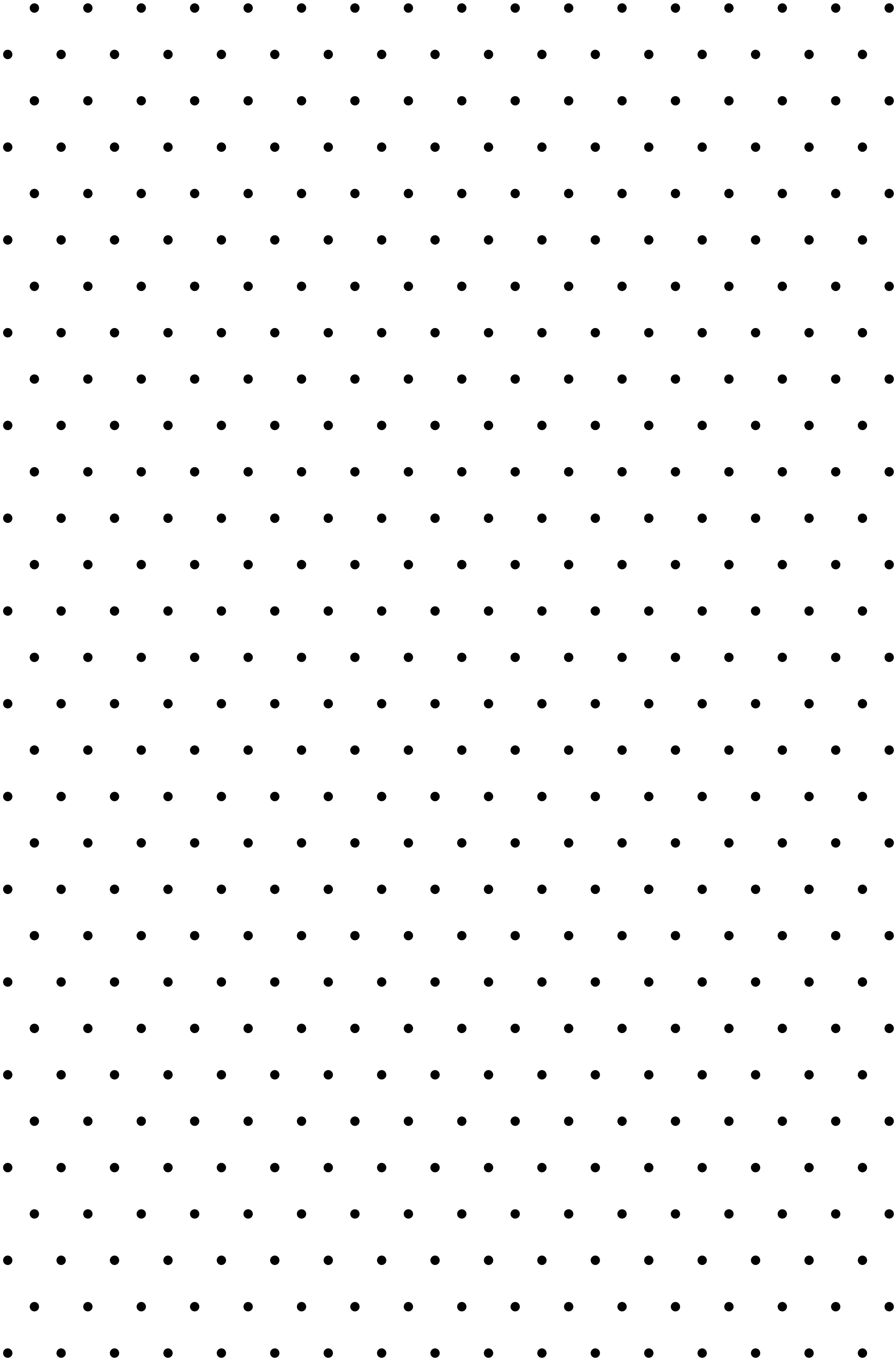
A **crossover** is a point where 2 lines cross. 	A **blob** is an area with lines around the outside and nothing in it. 	An **arc** is a line joining one crossover to another. 

Look at the shapes below.
Complete the pattern by drawing the loops in boxes 4 to 9.
For each shape, count the number of crossovers, blobs and arcs.

Write your results in the boxes underneath. Number 1 has been done already.

1			2			3		
								
a	*b*	*c*	*a*	*b*	*c*	*a*	*b*	*c*
2	2	1	4					
4			**5**			**6**		
a	*b*	*c*	*a*	*b*	*c*	*a*	*b*	*c*
7			**8**			**9**		
a	*b*	*c*	*a*	*b*	*c*	*a*	*b*	*c*

What do you notice from your results?
Can you explain the pattern in the numbers?



Points and Regions (Finding a Possible Formula)

n = number of points $\qquad$ r = number of regions

It turns out that the 4th differences are all 1, so the highest term in n must be $\frac{1}{24}n^4$. Using a spreadsheet we can calculate this amount and subtract it from r. Then we work out the new 1st, 2nd and 3rd differences. This time the 3rd ones are constant, and that gives us the term in n^3.

n	r	$\frac{1}{24}n^4$	$r-\frac{1}{24}n^4$	1st diff	2nd diff	3rd diff	$-\frac{1}{4}n^3$
1	1	0.042	0.958				–0.250
2	2	0.667	1.333	0.375			–2.000
3	4	3.375	0.625	–0.708	–1.083		–6.750
4	8	10.667	–2.667	–3.292	–2.583	–1.500	–16.000
5	16	26.042	–10.042	–7.375	–4.083	–1.500	–31.250
6	31	54.000	–23.000	–12.958	–5.583	–1.500	–54.000
7	57	100.042	–43.042	–20.042	–7.083	–1.500	–85.750
8	99	170.667	–71.667	–28.625	–8.583	–1.500	–128.000

Now we have the first two terms, we calculate $r-\frac{1}{24}n^4+\frac{1}{4}n^3$, which ought to be no more than quadratic. 2nd differences are constant at $1\frac{11}{12}$, so we have the term in n^2. Finally, subtracting this we obtain a linear sequence which we can write as $-\frac{3}{4}n+1$, and we've finished.

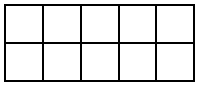
n	r	$r-\frac{1}{24}n^4+\frac{1}{4}n^3$	1st diff	2nd diff	$\frac{23}{24}n^2$	$r-\frac{1}{24}n^4+\frac{1}{4}n^3-\frac{23}{24}n^2$	1st diff	$-\frac{3}{4}n$	$r-\frac{1}{24}n^4+\frac{1}{4}n^3-\frac{23}{24}n^2+\frac{3}{4}n$
1	1	1.208			0.958	0.250		–0.750	1
2	2	3.333	2.125		3.833	–0.500	–0.750	–1.500	1
3	4	7.375	4.042	1.917	8.625	–1.250	–0.750	–2.250	1
4	8	13.333	5.958	1.917	15.333	–2.000	–0.750	–3.000	1
5	16	21.208	7.875	1.917	23.958	–2.750	–0.750	–3.750	1
6	31	31.000	9.792	1.917	34.500	–3.500	–0.750	–4.500	1
7	57	42.708	11.708	1.917	46.958	–4.250	–0.750	–5.250	1
8	99	56.333	13.625	1.917	61.333	–5.000	–0.750	–6.000	1

So we have $r=\frac{1}{24}n^4-\frac{1}{4}n^3+\frac{23}{24}n^2-\frac{3}{4}n+1$ or $r=\frac{1}{24}(n^4-6n^3+23n^2-18n+24)$. Using this formula we can predict values of r for larger n.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
r	1	2	4	8	16	31	57	99	163	256	386	562	794	1093	1471	1941	2517	3214	4048	5036

Rectangles in Grids

How many ways are there of fitting an $a \times b$ rectangle into an $x \times y$ grid, where a, b, x and y are all positive integers, and $\max(a,b) \le \min(x,y)$?
Every vertex of the rectangle must lie on a grid point, and the sides of the rectangle must be parallel to the grid lines.



- Start with rectangles that are 2×1.
  How many ways can you fit these rectangles into an $x \times y$ grid?

  Think about where the middles of the rectangles will go.
  There will be y rows of $x-1$ "horizontal" rectangles.
  There will be x columns of $y-1$ "vertical" rectangles.
  So the total number of rectangles, n, will be $n = y(x-1) + x(y-1) = 2xy - x - y$.
  e.g., for $x \times 2$ grids, $n = 3x - 2$;
  for $x \times 3$ grids, $n = 5x - 3$; etc.

- Now try rectangles that are $a \times 1$.
  Similar reasoning gives $n = y(x-a+1) + x(y-a+1) = 2xy - ax - ay + x + y$.
  This works provided $a \ne 1$.

- Now try rectangles that are $a \times b$.
  This time,
  $$\begin{aligned} n &= (y-b+1)(x-a+1) + (x-b+1)(y-a+1) \\ &= 2xy - ax - bx - ay - by + 2ab - 2a - 2b + x + y + 2 \end{aligned}$$, assuming $a \ne b$.

It's possible to extend this investigation to 3 dimensions, where you're fitting little cuboids into big cuboids (lattices).

The number of ways of fitting an $a \times b \times c$ cuboid into an $x \times y \times z$ cuboid is
$$\begin{aligned} n = &(z-c+1)(y-b+1)(x-a+1) \\ &+(z-c+1)(y-a+1)(x-b+1) \\ &+(z-b+1)(y-a+1)(x-c+1) \\ &+(z-b+1)(y-c+1)(x-a+1) \\ &+(z-a+1)(y-b+1)(x-c+1) \\ &+(z-a+1)(y-c+1)(x-b+1) \end{aligned}$$

This assumes that $\max(a,b,c) \le \min(x,y,z)$ and that none of a, b and c are equal to each other (otherwise symmetry will mean that there are fewer ways).

Stopping Distances

These are average stopping distances for an ordinary car on a good road surface.

speed (mph)	**thinking distance (m)**	**braking distance (m)**	**overall stopping distance (m)**
20	6	8	14
30	9	14	23
40	12	24	36
50	15	38	53
60	18	56	74
70	21	78	99

Answer these questions.

1. *Thinking distance* is the time it takes for the driver to realise he/she needs to press the brake pedal and to get his/her foot onto the pedal.
It's proportional to speed, so it doubles as the speed doubles.
Why do you think that is?

2. What "reaction time" do these numbers assume that the driver has?
Would you react that quickly?

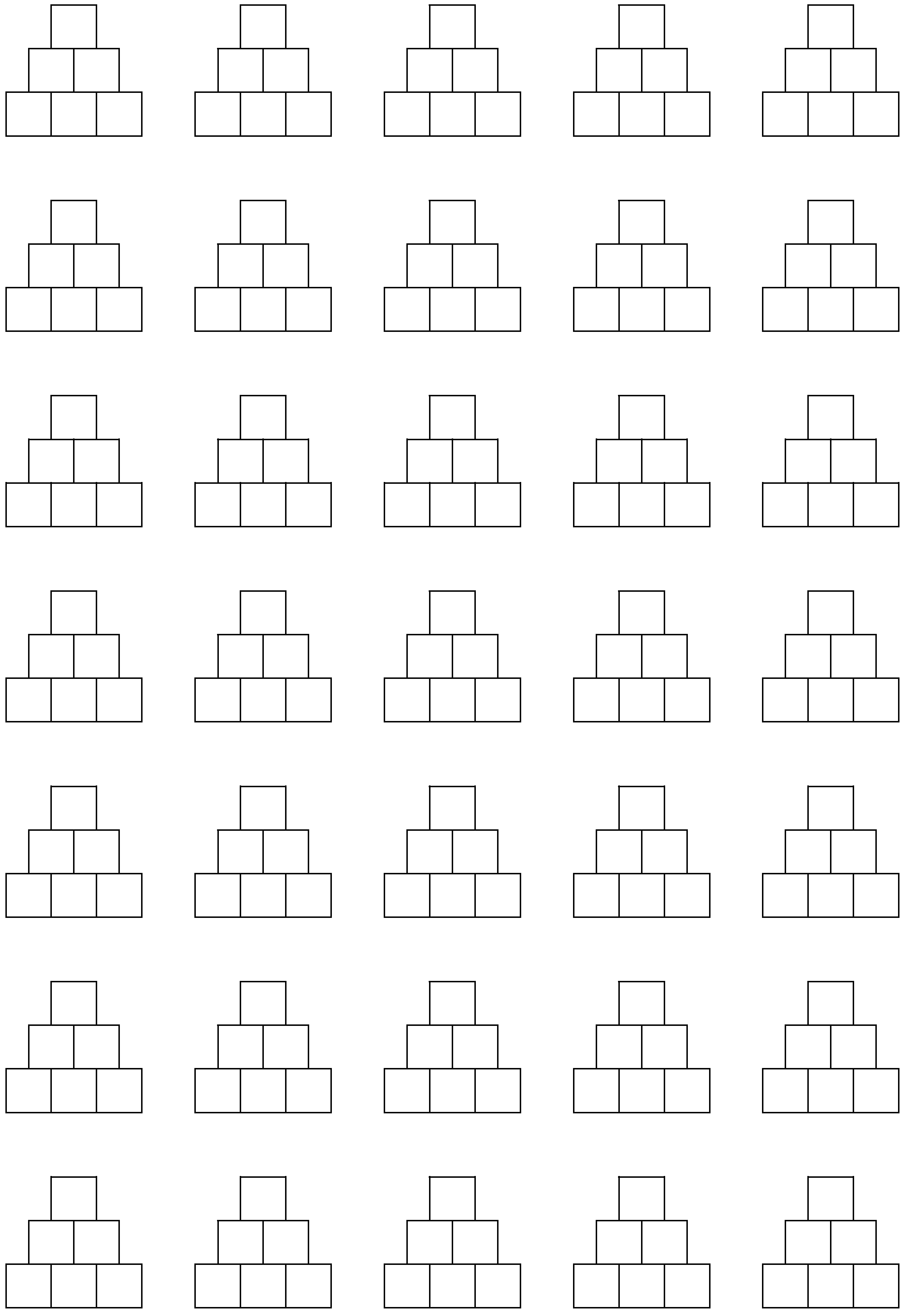
3. *Braking distance isn't* proportional to speed.
What happens to braking distance as the speed goes up?
Why do you think that is?

4. Find formulas for
- the *thinking distance* in terms of the speed;
- the *braking distance* in terms of the speed;
- the *overall stopping distance* in terms of the speed.

5. Some people think that the speed limit on motorways and dual carriageways should be raised to 80 mph.
Use your formula to predict the overall stopping distance at 80 mph.

6. How much more do you think these figures would be in wet or icy conditions?

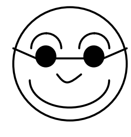
7. In real life, what factors apart from speed and weather conditions will affect the overall stopping distance?



1.20 Formulas, Equations, Expressions and Identities

- **Collecting terms** is equivalent to noting that 4 + 4 + 4 + 4 + 4 + 4 can be written as 6 × 4; i.e., that multiplication is repeated addition. It's wise to keep going back to numbers. Algebra is just generalised numbers: if it works with numbers, it works with algebra.
  Key questions are "Do numbers do that?", "Is that how numbers behave?"
  This is probably more helpful than treating letters as "objects"; e.g., with $3x+5x$ rather than asking "how many x's have we got altogether?" ask "how many *lots of* x have we got altogether?"
- Part of the problem of **simplifying** is knowing what counts as "simpler". Need to know what's impossible; e.g., that you can't simplify a^2+3b (you can't combine "unlike" terms). Where indices are involved, it sometimes helps temporarily to put the × signs back in; e.g., to simplify $3a^2b\times 4ab^3$, rewrite it as $3\times a\times a\times b\times 4\times a\times b\times b\times b$ (or imagine this), then remember that when multiplying you can change the order without it affecting the answer, so this equals $3\times 4\times a\times a\times a\times b\times b\times b\times b=12a^3b^4$.
- **Expanding** – getting rid of the brackets. For expanding pairs of binomials, some pupils use the **FOIL** acronym as a reminder that you get four terms: **f**ront, **o**utside, **i**nside and **l**ast. One way of illustrating this is to draw curves to link both terms in the *first* bracket with both terms in the *other* bracket.

$$(a+b)(c+d)$$



  And this gives a "smiley face", which is easy to refer to: e.g., "Have you done the nose?"
- **Factorising** – Just as with numbers, this means finding something that goes into it, although in algebra it can be "whole expressions" rather than whole numbers. So this is the opposite of expanding: "Now we've done all that work getting rid of the brackets, we want to work out how we can put them back in again!"
- **Balancing** – "ALWAYS DO THE SAME THING TO BOTH SIDES" makes a good title for a lesson! This is probably clearer than "moving" things from one side to the other, because it seems confusing that a positive term on the left becomes negative on the right when the two sides are supposed to be equal to each other.
- **Rearranging formulas** – This is a bit like solving equations; trying to find x (to make it the subject), say, in terms of the other quantities rather than as a numerical answer. It's helpful, again, to begin numerically with something like $24=2\times 10+4$ and to ask "Can we write an expression for 10 using the other numbers?": $10=\frac{24-4}{2}$.

1.20.1	Words that mean different things in maths from what they mean in ordinary life or in other subjects. Think of some examples. *Experimental constants are of a different kind.*	*Take-away, volume, factor, negative, net, etc. Also expression and identity.* *Discuss the difference between an unknown, a variable and a constant. (There are different sorts of "constants"; e.g.,* π *is absolute;* $g=9.81$ *m/s²* *varies from place to place.)*
1.20.2	Which do you think ought to be larger 456 × 458 or 457 × 457? What about other numbers?	*Answer: the second one (by 1), because* $(n+1)(n-1)\equiv n^2-1$. *Works even for negative numbers as well.*
1.20.3	**NEED** "Alphabet Code" sheets. A code like this one is probably better than using $a=1$, $b=2$, $c=3$, … (although that is easier to remember) because it avoids pupils getting the idea that those letters always stand for those numbers.	*Answers: 1. "Well Done"; 2. "I Love Algebra"; 3. "Maths is Fun"; 4. "Fish and Chips".* *Pupils can make up their own coded messages for each other using the same letter code. ("Remember I know the code, so don't say anything you wouldn't want me to read!")*

1.20.4 Balancing. **NEED** acetate (see pages) of scales and little weights and objects to slide around to create different problems.
"I have a weight problem!"

Pupils come to the projector to solve the problem by "always doing the same thing to both sides". They take off weights from both sides – eventually you can see what the mystery object must weigh.

Key questions are: "What are you going to do to both sides?", "What do you need to do to the left side?", etc.

kg is included in the "weights" so that the value of an object (its weight) isn't confused with the number of objects; i.e., the value of x isn't confused with the co-efficient of x.

The numbers are really mass *(amount of matter) rather than* weight *(heaviness), but this is an unnecessary complication here.*

It may be wise initially to praise any sequence of steps for solving a linear equation even if it isn't the shortest or most elegant. So long as you're doing the same thing to both sides each time, strategy can develop later.

It's probably worth setting out equation-solving very carefully: only one equals sign per line; all equals signs lined up vertically; curly arrows down the right side (or both sides) saying what happens between each step and the next; e.g.,

$$\begin{aligned} 12-2x &= x+3 && \quad +2x \\ 12 &= 3x+3 && \quad -3 \\ 9 &= 3x && \quad \div 3 \\ 3 &= x \end{aligned}$$

(Don't include kg in the equation-solving itself.) It's better to have curly arrows between *lines so that they don't get muddled up with the algebra* on *the lines.*

1.20.5 The numbers 2 and 2 have the strange property that 2 + 2 = 2 × 2. Find another pair of equal numbers that do the same thing.

Answer: $ab = a+b$ means that $a(b-1) = b$ and $a = \frac{b}{b-1}$. Clearly $a = 2$, $b = 2$ satisfy this, as do $a = 0$, $b = 0$.

What if the numbers don't have to be the same as each other?
What numbers are possible?
(The numbers don't have to be integers.)
What if 3 is one number – what does the other number have to be to make this work?

If they don't have to be equal, then there are infinitely many pairs of numbers that will work. The only number that neither can ever be is 1. Either both numbers are > 1 or else one number is < 0 and the other is between 0 and 1. (There is also the possibility that both numbers are 0, as mentioned above.) e.g., 3 and $1\frac{1}{2}$ or – 2 and $\frac{2}{3}$.

What about with three or more numbers?
When does sum = product?

With three numbers, the only integers that work are 1, 2, 3.

1.20.6 Painted cube.
A large cube is made out of 125 small cubes. The large cube is painted on the outside. When it's dismantled and the small cubes are examined separately, how many have paint on 3 sides, 2 sides, 1 side and no sides?
Do any cubes have more than 3 sides painted?
What about an $n \times n \times n$ cube?

Answers: no cube has paint on more than 3 sides.

sides painted	***5 × 5 × 5 cube***	***$n \times n \times n$ cube***	
0	*27*	$(n-2)^3$	*[the ones inside]*
1	*54*	$6(n-2)^2$	*[6 faces]*
2	*36*	$12(n-2)$	*[12 edges]*
3	*8*	8	*[8 corner cubes]*
total	*125*	n^3	

Total of right column

$$\begin{aligned} &= 8+12(n-2)+6(n-2)^2+(n-2)^3 \\ &= 8+(n-2)(12+(n-2)(6+(n-2))) \\ &= 8+(n-2)(12+(n-2)(n+4)) \\ &= 8+(n-2)(12+(n^2+2n-8)) \\ &= 8+(n-2)(n^2+2n+4) \\ &= 8+n^3+2n^2+4n-2n^2-4n-8 \\ &= n^3 \end{aligned}$$

It's a good algebra exercise to check that the total of the right column is n^3 (see working on the left).

1.20.7 What famous formulas do you know or have you heard of, perhaps in school?

They will most likely come from science.

Answers: (some possibilities)

- $F = ma$ *: Newton's (1642-1727) 2nd Law, where* F *is the resultant force in a particular direction acting on a particle of mass* m *and producing an acceleration of magnitude* a *in the same direction;*
- $E = mc^2$ *: Einstein's (1879-1955) famous equation from Relativity Theory, stating the equivalence of energy* E *and mass* m *, where* c *is the speed of light;*

Because c^2 *is so large, if even a tiny mass of matter is converted into energy, a huge amount of energy is released.*

- $$\frac{-h^2}{8\pi^2 m}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}\right)+V\psi=E\psi$$
  (Schrödinger's (1887-1961) equation from Quantum Mechanics for the wave function ψ *for a particle of mass* m *and energy* E *moving in a potential of* V *, where* h *is Planck's (1858-1947) constant (6.626 ×* 10^{-34} *J s).*

The chance of finding the particle at a particular spot is proportional to ψ^2 *at that spot (that's the meaning of* ψ *).*

1.20.8 Which formulas do you know that are interesting or useful to you?

Examples may come from hobbies, cooking, walking, thunder and lightning, etc.

A teacher's one is
concentration span (mins) = age (years) + 2

Do they agree with this one?
("Sorry, what were you saying?"!)

1.20.9 **NEED** Cards (see two sheets – back-to-back photocopy onto coloured card that is thick enough for the numbers not to show through). Hold up a letter card (e.g., f) facing the class. Look at the number on the back and say (for example) " $4f$ minus 20 equals 100" for pupils to work out mentally what f must be.

If pupils seem to be doing too well, they can probably somehow see the number on the back of the card!

Can play in teams.

Answer: 30

1.20.10 Hundred-Square Investigations.
Make a shape (e.g., a rectangle or a letter such as CEFHILOPTUX) out of squares and place it onto a 10 × 10 grid containing the integers from 1 to 100.
(See suitable A4 grid for an acetate; in section 1.16 there are smaller ones suitable for photocopying and guillotining for pupils to write on.)
Invent a rule (e.g., "add up all the numbers in the shape"; or "multiply the number in the bottom left square with the number in the top right square"; etc.) and investigate how that value changes when the shape is placed in different positions on the grid.
(It may be convenient to cut out the relevant shape from coloured acetate, so that you can slide it around on an OHP.)

Results can be proved by generalising the situation; e.g., for a U-shape,

x	$x+1$	$x+2$
$x+10$	$x+11$	$x+12$
$x+20$	$x+21$	$x+22$

It's best to decide on a position in the shape (e.g., the top left square) as the "reference number" to locate where the shape is placed on the grid.

So if the rule is "add up the left column and subtract it from the total of the right column and multiply the answer by the bottom middle square", then the result will be

$$(3x+36-(3x+30))(x+21)$$
$$=6(x+21)$$;

What if we allow the shape to rotate 90°?

i.e., 6 times the bottom middle number.

Alphabet Code

a	b	c	d	e	f	g	h	i	j	k	l	m
9	11	18	2	15	1	20	14	23	6	10	17	26
n	**o**	**p**	**q**	**r**	**s**	**t**	**u**	**v**	**w**	**x**	**y**	**z**
8	19	16	24	4	13	21	3	25	5	22	7	12

Work these out and decode the message.

A

1 $k-5$

2 $3w$

3 $g-u$

4 $6u-1$

5 $m-q$

6 $b+n$

7 $\frac{q}{u}$

8 $5r-5$

B

1 $2a+n$

2 $j+3$

3 $3y$

4 dy

5 $c-w$

6 $2k+3$

7 $w+2r$

8 $2s-v$

9 $\frac{e}{5}$

10 $\dfrac{m-d}{u}$

C

1 $b+6d$

2 $\dfrac{t+z+1}{2}$

3 $c+1$

4 $a+d+h$

5 $\dfrac{30}{d}$

6 $\dfrac{c}{1+f}$

7 $a+\dfrac{p}{2}$

8 wr

9 fwu

10 $x-11$

11 $\dfrac{dr}{2}$

12 $2h-c+1$

D

1 $\dfrac{j}{d+r}$

2 $m-3$

3 $2r+w$

4 $2y$

5 $\dfrac{c}{d}$

6 $p-n$

7 $\dfrac{x}{1+k}$

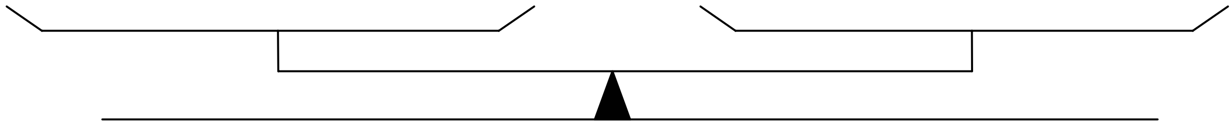
8 $2f+p$

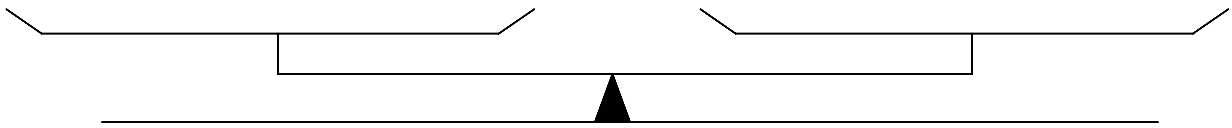
9 $s+f$

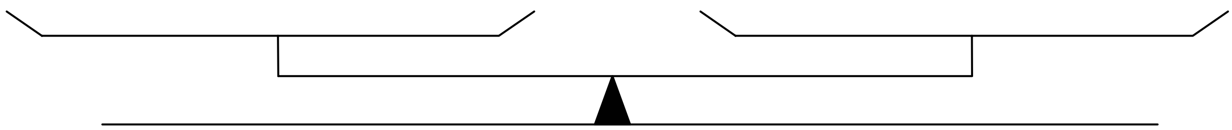
10 $2e-y$

11 $n+\dfrac{p}{2}$

12 $u+4f+n-2$

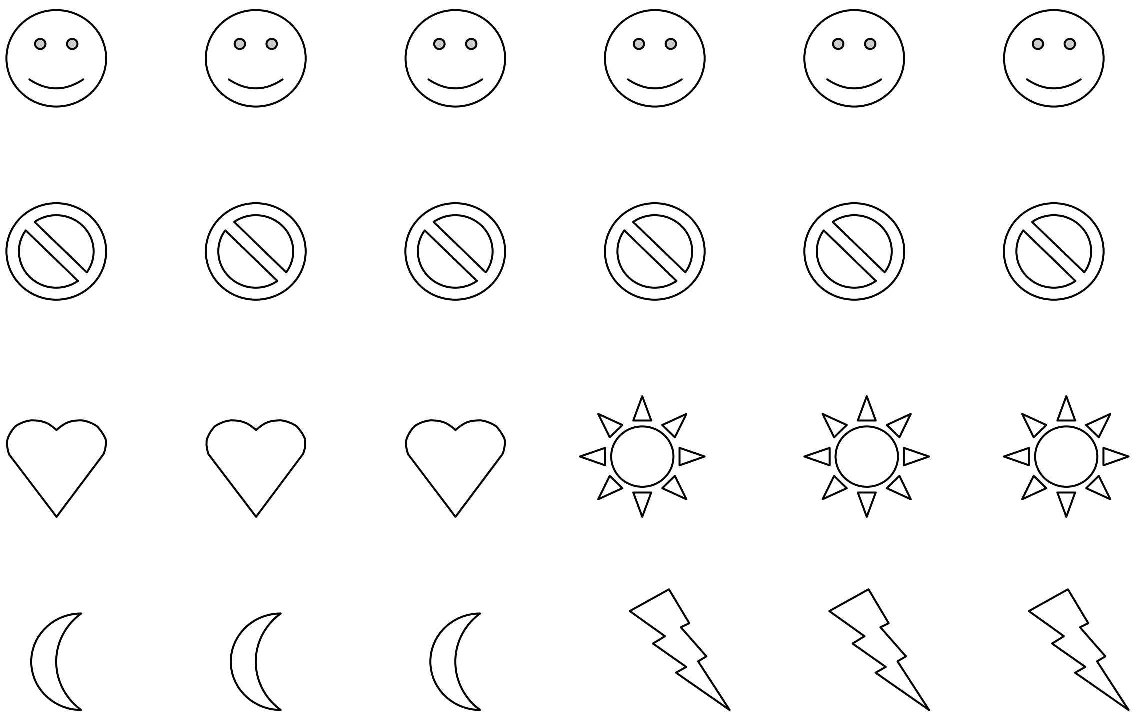






Photocopy onto acetate and then cut out the shapes.
They're easier to use if you leave a gap of 0.5 cm or so around the edge. Keep them in two envelopes.

Shapes to represent unknowns (put in one envelope)



Weights (put in another envelope)

1 kg	1 kg	1 kg	2 kg	2 kg
3 kg	3 kg	3 kg	2 kg	2 kg
4 kg	4 kg	4 kg	5 kg	5 kg
10 kg	12 kg	15 kg	5 kg	5 kg
10 kg	12 kg	20 kg	6 kg	6 kg

Sheet 1 of 2 Photocopy the two sheets back-to-back onto coloured card and then guillotine. The card must be thick enough for the numbers not to show through.

a	*b*	*c*	*d*
e	*f*	*g*	*h*
i	*j*	*k*	*m*
n	*p*	*q*	*r*
s	*t*	*u*	*v*

 Photocopy the two sheets back-to-back onto coloured card and then guillotine.
The card must be thick enough for the numbers not to show through.

10	3	7	24
13	2	30	1.5
66	9	5	0
73	1	0.1	55
8	3	–2	34

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

1.21 Quadratic Equations

- One way to begin is by "trying" as a class exercise together to solve an equation such as $x^2 + x = 25$ by simple manipulation. Unless pupils see that this is hard, they will wonder why we need some special approach to "quadratic equations". It's only when you try to divide by x, say, and end up with terms like $\frac{25}{x}$ getting in the way, that you realise that it isn't straightforward. So you could just ask for suggestions and do to both sides *exactly* what is suggested.

1.21.1 One way to multiply two binomials is to use boxes.
e.g., to expand $(x+3)(x-2)$, write

x	3	
x^2	$3x$	x
$-2x$	-6	-2

So $(x+3)(x-2) = x^2 + 3x - 2x - 6 = x^2 + x - 6$.
So $(x+3)(x-2) = 0$ and $x^2 + x - 6 = 0$ are the same equation.
But the factorised version is easier to solve.

So we need to be able to go backwards from $x^2 + x - 6$, using the boxes, to get $(x+3)(x-2)$.

We write

x		
x^2		x
	-6	

and then try to find numbers to place on each side that multiply to make –6 and have a sum of 1.

It may be worth doing some number work finding pairs of numbers with sums/differences of a certain amount that multiply to give a certain amount, especially involving negative numbers. (See "Number Puzzles" sheet.)
Answers to "Number Puzzles":

question (sum)		***question (difference)***	
1	*2, 8*	*1*	*10, 20*
2	*4, 6*	*2*	*1, 11*
3	*2, 10*	*3*	*2, 10*
4	*1, 11*	*4*	*4, 12*
5	*5, 7*	*5*	*1, 9*
6	*10,10*	*6*	*12, 20*
7	*9, 11*	*7*	*50, 100*
8	*4, 16*	*8*	*30, 50*
9	*6, 10*	*9*	*70. 90*
10	*4, 12*	*10*	*2, 22*
11	*5, 11*		
12	*2, 15*		
13	*5, 12*		
14	*15, 20*		
15	*10, 25*		

You can use instinct or be systematic and write out the pairs of factors of –6 (in this context it's possible to talk about the factors of negative integers): (1, –6); (–1, 6); (2, –3); (–2, 3).

1.21.2 Completing the square or using the formula.
If $ax^2 + bx + c = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, but this formula "goes wrong" if $b^2 < 4ac$ and you don't get any "real" solutions.

You could check answers from previous trial and improvement work – we can now solve those problems much more easily.
If $b^2 > 4ac$, then you get two solutions; if $b^2 = 4ac$ then you get just one solution.

1.21.3 What about when the co-efficient of x^2 isn't equal to 1? If you start with $(rx+s)(tx+u)$ and expand the brackets you get $rtx^2 + (ru+st)x + su$, so a trick for factorising $ax^2 + bx + c$ is to multiply the constant term ($c = su$) by the co-efficient of x^2 ($a = rt$) to get $rstu$. Then you look for two numbers which multiply to make this much and have a sum the same as the co-efficient of x.

For example, to factorise $6x^2 + 11x - 10$, work out $6 \times -10 = -60$ and look for two numbers that have a product of –60 and a sum of 11: they are 15 and –4. So we write $6x^2 + 15x - 4x - 10$, and then factorise this into $3x(2x+5) - 2(2x+5)$ $= (3x-2)(2x+5)$.

This is usually easier than the various other possible methods.

Number Puzzles

We are two numbers.
We add up to 10 and our product is 21.
What are we?   Answer: 3 and 7.

Now try these.

1 We add up to 10 and our product is 16.
2 We add up to 10 and our product is 24.
3 We add up to 12 and our product is 20.
4 We add up to 12 and our product is 11.
5 We add up to 12 and our product is 35.
6 We add up to 20 and our product is 100.
7 We add up to 20 and our product is 99.
8 We add up to 20 and our product is 64.
9 We add up to 16 and our product is 60.
10 We add up to 16 and our product is 48.
11 We add up to 16 and our product is 55.
12 We add up to 17 and our product is 30.
13 We add up to 17 and our product is 60.
14 We add up to 35 and our product is 300.
15 We add up to 35 and our product is 250.

We are two numbers.
This time our *difference* is 10 and our product is 24.
What are we?   Answer: 2 and 12.

Now try these.

1 Our difference is 10 and our product is 200.
2 Our difference is 10 and our product is 11.
3 Our difference is 8 and our product is 20.
4 Our difference is 8 and our product is 48.
5 Our difference is 8 and our product is 9.
6 Our difference is 8 and our product is 240.
7 Our difference is 50 and our product is 5 000.
8 Our difference is 20 and our product is 1 500.
9 Our difference is 20 and our product is 6 300.
10 Our difference is 20 and our product is 44.

Number Puzzles

We are two numbers.
We add up to 10 and our product is 21.
What are we?   Answer: 3 and 7.

Now try these.

1 We add up to 10 and our product is 16.
2 We add up to 10 and our product is 24.
3 We add up to 12 and our product is 20.
4 We add up to 12 and our product is 11.
5 We add up to 12 and our product is 35.
6 We add up to 20 and our product is 100.
7 We add up to 20 and our product is 99.
8 We add up to 20 and our product is 64.
9 We add up to 16 and our product is 60.
10 We add up to 16 and our product is 48.
11 We add up to 16 and our product is 55.
12 We add up to 17 and our product is 30.
13 We add up to 17 and our product is 60.
14 We add up to 35 and our product is 300.
15 We add up to 35 and our product is 250.

We are two numbers.
This time our *difference* is 10 and our product is 24.
What are we?   Answer: 2 and 12.

Now try these.

1 Our difference is 10 and our product is 200.
2 Our difference is 10 and our product is 11.
3 Our difference is 8 and our product is 20.
4 Our difference is 8 and our product is 48.
5 Our difference is 8 and our product is 9.
6 Our difference is 8 and our product is 240.
7 Our difference is 50 and our product is 5 000.
8 Our difference is 20 and our product is 1 500.
9 Our difference is 20 and our product is 6 300.
10 Our difference is 20 and our product is 44.

1.22 Simultaneous Equations

- The main difficulty is with the concept of two (or more) constraints being simultaneously true. Many puzzles involve this type of situation (see below), so beginning with some of these helps with seeing the nature and usefulness of this kind of problem.
- The **substitution** method is simplest when both (we'll assume that there are just two) equations are given as $y =$ something. "If $y =$ *this* and $y =$ *that* then *this* = *that*." "If $a = c$ and $b = c$ then $a = b$".
  The substitution method involves rearranging one equation so that we get an expression for one of the unknowns and then substitution this expression into the other equation.
- **Elimination** is usually more elegant but takes some getting used to.
  Can we really multiply an *equation* (rather than just a *term* or a *side*) by 2, and why doesn't it seem to get twice as big? In fact it's still the "same" equation afterwards. Can we really add two *equations* together? Should we even expect to get an equation as a result? (If we "added" 2 triangles together that wouldn't make another triangle.) Pupils need to see that the 1st equation is telling us that LHS = RHS (they're the same amount), so when we "add up two equations" we're really just adding the same amount to both sides of the 2nd equation: we add the LHS to one side and the RHS to the other, but since LHS = RHS we're really adding the same thing to both sides. When we "multiply an equation by 3", say, we can think of it as "scaling up the scales" – multiplying the weight of everything on both sides of the scales by 3 will still leave it balancing.
- Seeing the connection with intersecting graphs is very helpful, particularly for seeing when either there is no solution (e.g., $x+y=4$ and $x+y=5$, parallel lines never cross) or infinitely many pairs of (x, y) values work (e.g., $x+y=4$ and $2x=8-2y$, equations of the same line).

1.22.1 Can start by trying to "solve" an equation like $x+y=10$ or an equivalent problem: "I'm thinking of two numbers. When I add them together I get 10. What are the numbers?"

There are infinitely many pairs of solutions, not just positive integer but decimal and negative too.

Pairs of solutions (x, y) all lie on the graph of the equation $x+y=10$.

To get just one pair of answers, we need an extra condition; e.g., "y is 4 bigger than x" – how can we write that? $y = x+4$
Substitute the 2nd equation into 1st one and solve. Or draw the 2nd graph and see where it crosses the 1st one.

Answer: $x=3$ and $y=7$.

1.22.2 "Fruit machine"–type puzzles (see sheet – suitable for putting on acetate).

Pupils can bring in examples of this type of puzzle from newspapers/books/magazines to see if we can solve them more quickly using algebra. *(Sometimes it's faster to use intuition; sometimes it's faster to use algebra.)*

Answers:
$f=3$; $h=4$; $c=5$; $t=1$ *(face, heart, etc.);*
$s=4$; $c=0$; $d=8$; $h=2$ *(spade, club, etc.);*
$p=12$; $e=4$; $y=5$; $d=11$ *(pound, euro, etc.);*
f *(face)* $=9$; a *(arrow)* $=10$; n *(quaver)* $=4$; m *(2 quavers)* $=6$;
$u=6$; $d=3$; $r=1$; $l=8$ *(up, down, etc.);*
$a=3$; $b=1$; $c=10$; $d=5$ *(letters).*

1.22.3 Alex has £160 in a bank account and he deposits £8 per week. His sister Becky has £270 in her bank account but she withdraws £3 every week. If they carry on like this, how many weeks will it be before they have the same amount as each other in their bank accounts?

Let $t =$ number of weeks.
Then $160+8t = 270-3t$, so $t=10$ weeks.

Or "Becky starts off with £110 more than Alex and at the end of every week Alex ends up £11 better off than Becky (£8 + £3), so it will take 10 weeks before they're equal."

1.22.4 Circus acrobatics.
A circus family perform acrobatics with the children standing on their parents' shoulders. All the sons are the same height as each other. The father and his son make a height of 9 ft, and the father and 2 sons make 12 ft. How tall are the man and the sons?

Diagrams make it clear what's going on with the algebra. (Draw a line down the middle of the page and do the algebra on the left and the corresponding drawing – stick people – on the right.)

$f + s = 9$; $f + 2s = 12$; *so* $f = 6$ *ft and* $s = 3$ *ft.*

Consider a different family (with different heights). This time if the son stands on the father's shoulders the total height is 7 ft, but if the father holds his son by the ankles, the distance from the ground to the top of the son's head is 3 ft. How tall are this father and his son?

This time, $f + s = 7$; $f - s = 3$; *so* $f = 5$ *ft and* $s = 2$ *ft.*

Pupils can make up their own situations involving acrobats on horseback, stilts, etc.

To make things more complicated, you can introduce a mother (height m) and daughters (height d). Pupils can make up their own puzzles along these lines.

e.g., the mother stands on the father's shoulders holding a son by his ankles who holds his sister by her ankles, etc.

If all four "sizes" of people are involved, how many positions would they have to get into for us to be able to work out their heights with certainty?

For 4 unknowns, you need at least 4 equations.

If they hang over the edge of a drop, then you can have "total heights" which are negative.

1.22.5 Solve these equations simultaneously.

$ab = 1$
$bc = 2$
$cd = 3$
$de = 4$
$ea = 6$

(Note that none of the equations has a right side of 5.)

Pupils can make up similar sets of complicated-looking simultaneous equations for each other to solve.
Pick what the numbers are going to be first!

There are many methods of solution. One way is to multiply all the equations together to give $(abcde)^2 = 1 \times 2 \times 3 \times 4 \times 6 = 144$,
so $abcde =$ *12 or –12.*
Multiplying the 1st and the 3rd gives $abcd = 3$, *so* $e = \pm 4$. *Therefore* $a = \frac{6}{4} = 1\frac{1}{2}$, $b = \frac{1}{1\frac{1}{2}} = \frac{2}{3}$,

$c = \frac{2}{\left(\frac{2}{3}\right)} = 3$, $d = 1$ *and* $e = 4$, *or all of* a *to* e *could be negative of those values.*

1.22.6 David is twice as old as Henry was when David was as old as Henry is now. If the sum of their ages is 49 years, how old is David?

(Start by working out who is older, Henry or David.)

Answer: 28 years old.
Clearly David is older than Henry, since at some time in the past David was Henry's current age.
There are only 2 unknowns: David's age now (d) *and Henry's age now* (h)*, since the time elapsed since David was as old as Henry is now is just* $d - h$ *years.*
Therefore, $d + h = 49$; *and*
$d = 2 \times (h - (d - h)) = 2(2h - d)$, *so* $3d = 4h$, *so*
$h = 21$ *and* $d = 28$ *years old.*

1.22.7 There are a certain (integer!) number of rabbits and chickens in a cage. If altogether there are 23 heads and 62 legs, how many of each are there?
(Expect to be asked how many legs a chicken has!)

Answer: 8 rabbits, 15 chickens.
You could solve $r + c = 23$ *and* $4r + 2c = 62$ *simultaneously or imagine they were all chickens. That would make 2 × 23 = 46 legs, but there are 62 – 46 = 16 extra legs, and these must come from 8 rabbits. So 15 chickens.*

1.22.8 A mother agrees to pay her daughter £3 every night that she does her homework, provided that the *daughter* pays the *mother* £4 every night that she *doesn't* do any homework. (Would you agree to this arrangement?!) After 28 days, the daughter has paid her mother exactly the same amount of money that her mother paid *her* during that time. How many days' homework did the daughter do?

Answer: 16 days of homework.
Let h *= number of days that the daughter did her homework. Then she must have skipped her homework on* $28-h$ *days.*
Therefore she will have gained
$3h-4(28-h)=0$ *pounds altogether.*
Solving this gives the answer.

Alternatively, you can argue that the ratio of days of homework to days of no homework must be $4:3$ *so as to "cancel out" the* $3:4$ *ratio of payment. So we have to split 28 days into the ratio* $4:3$*, which gives* $16:12$*, so she must have done 16 days of homework.*

1.22.9 Helen works as a part-time waitress and gets paid £6 per hour. For overtime she gets £7 per hour. One week she worked for 12 hours and earned £76. How much overtime did she do?

Or assume that it was all at the basic rate. That would have earned her 12 × 6 = £72, but £76 is £4 more so she must have earned those £4 by doing 4 of the hours as overtime.

Answer: 4 hours
This can be tackled in a similar way to above or by letting $x=$ *number of hours of normal work and* $y=$ *number of hours of overtime, so that*

$$6x+7y=76$$
$$x+y=12$$

and then solving these equations simultaneously.

1.22.10 The formula $f=\frac{9}{5}c+32$ converts temperature in Celsius (c) to temperature in Fahrenheit (f). At what temperature are the values the same as each other?

Answer: –40 °C= –40 °F

$$c=\tfrac{9}{5}c+32$$
$$\tfrac{4}{5}c=-32$$
$$c=-40$$

At what temperature is one value twice as big as the other?

Answer: 2 possibilities:
Either $f=2c$*, so*

$$2c=\tfrac{9}{5}c+32$$
$$\tfrac{1}{5}c=32$$
$$c=160$$

and $f=320$*, so 160 ˚C and 320 ˚F;*

or $c=2f$*, so*

$$f=\tfrac{9}{5}(2f)+32$$
$$\tfrac{13}{5}f=-32$$
$$f=-12.31$$

and $c=-24.62$*, so –12.3˚F and –24.6˚C (both to 2 dp).*

Scientists often measure "absolute temperature" in *Kelvin*. The relationship between temperature in Kelvin (k) and temperature in Celsius (c) is approximately $k=c+273$. At what temperature will the values be the same as each other?

Never, because the equations $k=c+273$ *and* $k=c$ *are inconsistent and have no solution.*

When will the Kelvin temperature be the same as the Fahrenheit temperature?

If $k=f$, then

$$c+273=\tfrac{9}{5}c+32$$
$$241=\tfrac{4}{5}c$$
$$301.25=c$$

so at about 301 ˚C the Fahrenheit and Kelvin temperatures are both about 574 (K or ˚F).

Pupils can invent similar puzzles to these.

☺	☺	☺	☺	12
☺	♡	☺	♡	14
☺	△	♡	❒	13
♡	△	❒	△	11

♠	♠	♠	♠	16
♣	♠	♣	♠	8
♣	♥	♥	♠	8
♥	♠	♦	♣	14

£	€	¥	¥	26
€	£	£	£	40
$	$	£	£	46
¥	€	€	€	17

☺	☺	↑	♪	32
♫	♫	♫	♪	22
↑	↑	♫	♪	30
☺	♫	☺	↑	34

▲	▲	▼	▼	18
▲	▼	▶	▶	11
◀	◀	▶	▶	18
▲	▲	▲	▼	21

a	b	b	c	15
a	b	c	c	24
d	a	a	a	14
d	d	d	d	20

1.23 Co-ordinates and Straight-Line Graphs

- It's worth getting into the habit of always writing co-ordinates in (x, y) form with comma and brackets. Even in oral work you can ask pupils to say "bracket-number-comma-number-bracket" whenever they describe co-ordinates so as to reinforce that they're always given that way.
- Drawing axes is often difficult: things to emphasise include numbering the *lines,* not the spaces; labelling the x and y axes the right way round (the letter " y " has a "vertical" tail to remind you that it goes with the vertical axis); not missing out the lines next to the axes because the numbers are in the way; using the same scale on both axes (unlike in other graph work) so as not to distort angles, shapes and areas.
- For plotting co-ordinates accurately ("along the corridor and up the stairs" helps with the order; x then y is alphabetical order), it's helpful to get pupils up to the board to do this, so that common mistakes can be discussed and avoided. (If you don't have part of the board covered with squares, you could make an acetate out of the 1 cm × 1 cm squared paper page following and use an OHP.)
- It's important to be able to tell what is a linear equation (that will give a straight-line when you draw it) and what isn't. Only if it can be rearranged into $y = mx + c$ or $x = k$ will the graph be a straight line.

1.23.1 Co-ordinate Shapes/Pictures.
These are readily available in many books; e.g., Christmas designs, animals, etc.

Good for wall displays.

You can link this topic to names of quadrilaterals/polygons and kill two birds with one stone.
("Plot these points and join them up – name the resulting polygon.")
Pupils can make up their own.

You need to make sure that it's clear in what order the points must be plotted and whether pupils are to join up each point to the next with a straight line as they go. It doesn't generally work to plot all the points first and then try to join them up in the right order afterwards.

1.23.2 Equations of straight lines (see sheet of axes and thick line suitable for making into acetates).
You can draw on the acetate with washable projector pens or use the thick line page.
(It's good to make two copies of this because intersecting lines are useful; e.g., for simultaneous equations).

Projecting a grid onto a white board and then drawing on the whiteboard often goes wrong because if you bump the projector it's hard to get it back exactly where it was. Also, if there's any distortion of the image, a ruled line on the board won't appear straight against the grid. So it's best to put the line somehow onto the projector (e.g., using the "thick line" acetate given or a piece of wire) and slide it around. "What's the equation of this line?", etc.

Put up a line (e.g., $x = 4$) and ask pupils to tell you the co-ordinates of any of the points along that line. "What's the same about all those co-ordinates?" The first (x) number is always 4, so we call the line $x = 4$.
You can use a similar process on more complicated equations; e.g., $y = 2x + 3$.

Always start by picking out co-ordinates of several points on the line and looking for some connection between the x-numbers and the y-numbers. Later we can notice that in $y = mx + c$ the gradient is m and the intercept on the y-axis is $(0, c)$.

Tell me the equation of a line that would go through (2,7), etc.

e.g., $y = x + 5$, $y = 3x + 1$, $x = 2$, etc.

1.23.3 How can two lines have the same steepness but not be parallel?

Answer: their gradients are of equal size but opposite sign – one is minus *the other.
They're reflections of each other in the vertical axis; e.g., gradients 3 and –3.*

1.23.4 It's sensible to spend some time on gradient before using it with $y = mx$ and later with $y = mx + c$.

From one point on a straight line to another point on the same line,

$$\textbf{gradient} = \frac{\textbf{distance up}}{\textbf{distance to the right}}$$

Gradient doesn't have to be on co-ordinate axes. e.g., "Show me with your arm a gradient of 2."

"I'm driving along a road with gradient $\frac{1}{3}$ and then it changes to $-\frac{3}{4}$. Describe what's happened."

Going uphill to downhill, but the downhill is steeper than the uphill was.
Positive gradient means uphill (from left-to-right). Negative means downhill.

Use "thick line" acetate and squared acetate (without axes) to do quick questions: "What's the gradient now?"

If the gradient is the same all the way along a line it will be a straight line (that's what makes it straight).

Gradient number-line:

–1.0	–0.5	0	0.5	1.0
downhill, getting ← steeper		flat	uphill, getting steeper →	

A gradient of 1 (or –1) is 45° to the axes.

You can use the formula $m = \frac{\Delta y}{\Delta x} = \frac{y_B - y_A}{x_B - x_A}$ *for the gradient of the straight line joining* (x_A, y_A) *to* (x_B, y_B). *Pupils can think about why it works for negative co-ordinates as well as positive.*

1.23.5 People maths. Seat pupils in rectangular array with roughly equal distances separating pupils along the rows and down the columns. Set the pupil in the far back right corner (from the teacher's point-of-view) as the position (0,0) and then work along the rows and columns assigning each pupil a pair of co-ordinates:

Depending on your classroom layout, you may not be able to do this perfectly. Science labs with benches fixed to the floor are particularly awkward: you could decide to take pupils into the hall or outside.

(4,0)	(3,0)	(2,0)	(1,0)	(0,0)
(4,1)	(3,1)	(2,1)	(1,1)	(0,1)
(4,2)	(3,2)	(2,2)	(1,2)	(0,2)
(4,3)	(3,3)	(2,3)	(1,3)	(0,3)
(4,4)	(3,4)	(2,4)	(1,4)	(0,4)
(4,5)	(3,5)	(2,5)	(1,5)	(0,5)

TEACHER

It may come out different from this depending on your classroom and class size.

This way the xy *grid is the right way round for the class, and the teacher just has to keep alert!*

Go along rows and columns with each person reading out his/her co-ordinates.
"What's the same all the way along that row/column?" e.g., x values all 2.
"Stand up if your y value is one more than your x value." "Stand up if your x value is less than your y value" etc.
"Who hasn't stood up yet?" "What equation could we say that would involve you?"

These instructions give the graphs $x = 2$, $y = x + 1$ *and* $x < y$ *(a region this time, rather than just a line).*

1.23.6 You could investigate the relationship between the gradients of perpendicular lines by drawing, calculating, recording and looking for a pattern.
(You don't really need protractors if you use squared paper, or you can use the corner of a piece of paper to check for a right angle.)

Answer: perpendicular gradients multiply to make –1.

So if a line with gradient m_1 *is perpendicular to a line with gradient* m_2, *then* $m_1 m_2 = -1$.

Is it true that any two lines are either parallel or they intersect each other?

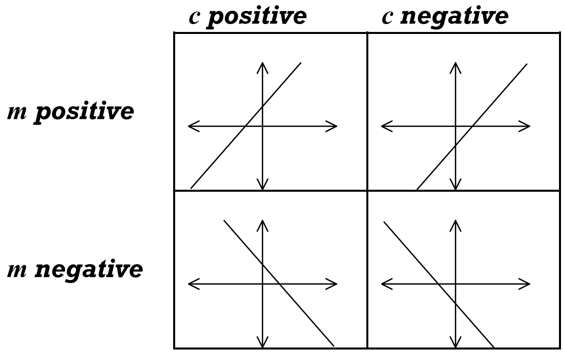
Yes in 2 dimensions, but in 3 dimensions it isn't because "skew lines" are non-parallel non-intersecting lines.

1.23.7 Use a computer graph-plotting program to draw lots of straight-line graphs and look at the effects of changing m or c (big/small, positive/negative).

Pupils could record their results visually in a table (see right).

In each case, the origin is at the point where the axes cross.

e.g., investigating positive/negative effect:



1.23.8 Open-Ended Questions:
1 Tell me the equations of 4 lines that go through (2,1).
2 Tell me the equations of 4 lines that make a square.
3 Tell me the equations of 4 lines that make a "tilted" square (one where the sides aren't horizontal and vertical).
4 Which of these points is the odd one out because it doesn't lie on the same straight line as the other three?
a (4, 11); b (–2, –5); c (0, –1); d (5, 14)
5 What's special about the equations of parallel lines? What about the equations of a pair of perpendicular lines?

Answers:
e.g., $y=\frac{1}{2}x$; $y=x-1$; $y=1$; $x=2$

lots of possibilities

b; all the others lie on the line $y=3x-1$

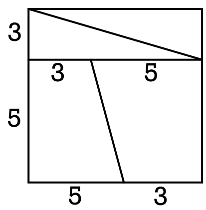
parallel lines have the same m *value; perpendicular have* m *values that multiply to make –1 (see section 1.23.6).*

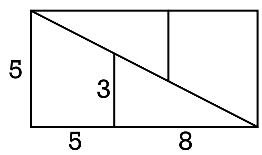
1.23.9 Battleships (play in pairs).
You need to use a fairly large grid to make it interesting; e.g., –10 to 10 for both x and y.
Each player picks 4 pairs of co-ordinates for his/her battleships, and they take it in turns to call out equations of straight lines (torpedoes) *which have to come from one of their own ships*. They draw the line and if it goes through a battleship it sinks it (or 2 hits to sink – teacher's discretion!). The player whose ships are all sunk first is the loser.
Pupils could keep a record of torpedo equations in case of disagreement afterwards!

One strategy is to use lines which go through the maximum number of points in the –10 to 10 range; e.g., $x+y=20$ *would be a poor choice, because it goes through (10,10) only.*

Also, where the paths of the enemy's torpedoes "cross" may be the location of one of his/her ships, but because he/she has 4 there is no guarantee!

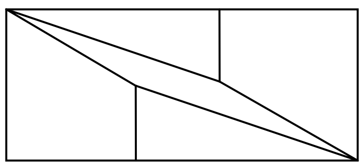
1.23.10 Dissection Puzzle.
Use 1 cm × 1 cm squared paper to draw an 8 cm × 8 cm square (area 64 cm^2).
Cut it up as shown below and rearrange it to make a 5 cm × 13 cm rectangle (area 65 cm^2).
Where has the extra 1 cm^2 come from?

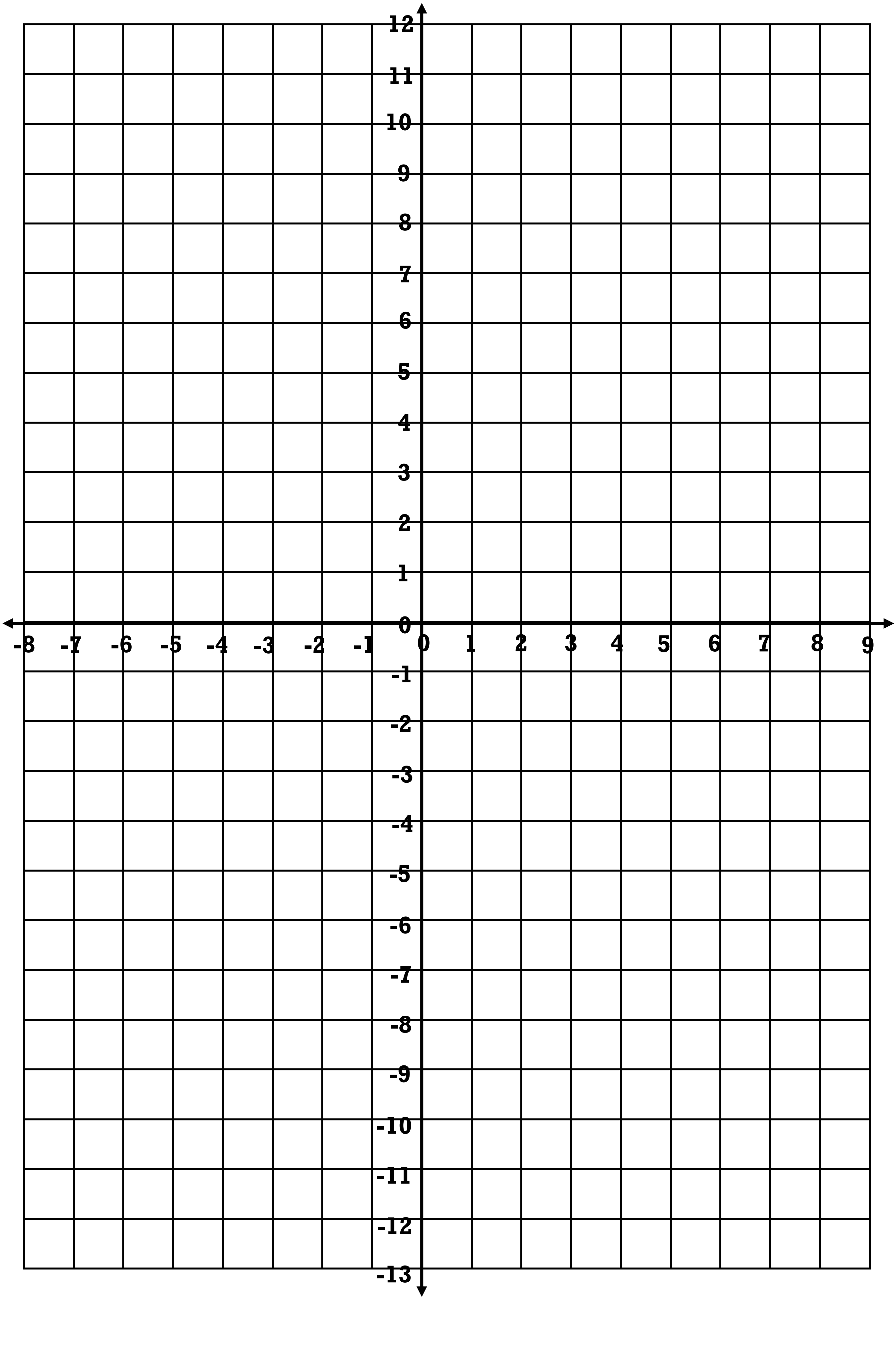




Answer: In the second arrangement, the gradient of the top part of the "diagonal line" is $-\frac{2}{5}$*, but the bottom part has gradient* $-\frac{3}{8}$*, so the "diagonal line" is in fact not a single straight line but two lines.*

The exaggerated drawing below shows that the missing 1 cm^2 is contained in the parallelogram gap left in the middle of the shape.




12
11
10
9
8
7
6
5
4
3
2
1
0
-1
-2
-3
-4
-5
-6
-7
-8
-9
-10
-11
-12
-13
-8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9

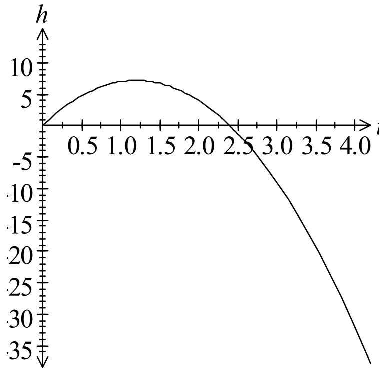
1.24 Polynomial Graphs

- Some pupils may think that all graphs that come from algebra are straight lines and that you can't get a curve from an algebra equation. It may be an eye-opener to see that maths can produce graphs that look like "real-life" graphs with all their bumps and turns. This makes for a good opportunity to discuss the principles of mathematical modelling.

1.24.1 Mathematical Modelling. Projectiles.
The equation $h = 12t - 5t^2$ models the motion of a stone thrown off the top of a cliff (ignoring air resistance, size and shape of stone, curvature of the earth, Einstein's theory of relativity, etc.).
h is the height of the stone in metres above the top of the cliff at time t seconds after letting go of it.
Draw a graph of h against t for the first 4 seconds (do a table of numbers first).

1. When does the stone reach the maximum height?
2. What is the maximum height?
3. When is the stone level with the cliff top?
4. The bottom of the cliff drop is 26 m. When does the ball land?
5. When is it going fastest?

Use 1 mm × 1 mm squared graph paper



Answers:
1. *After 1.2 s*
2. *7.2 m*
3. *After 2.4 s*
4. *After 3.78 s (2 dp)*
5. *As it's just about to hit the bottom (gradient has largest value – although negative); the vertical speed on impact is about 26 m/s.*

1.24.2 Pupils could investigate how the gradients of curved graphs vary as x varies.
Could try to draw the gradient functions on a sheet of different functions $f(x)$ (see sheet).

This could lead to differential calculus (see several sheets that do this with $y = x^2$ and $y = x^3$).

1.24.3 **NEED** a computer graph-plotting program.
Investigate the shape of the graphs of these functions with different numbers instead of the letters a and b.
Try a positive/negative and b positive/negative. Sketch the results in a table (see right) for each function.
Could split up the class so that different pupils/groups cover different functions.

$y = ax + b$

$y = ax^2 + b$

$y = a\sqrt{x} + b$

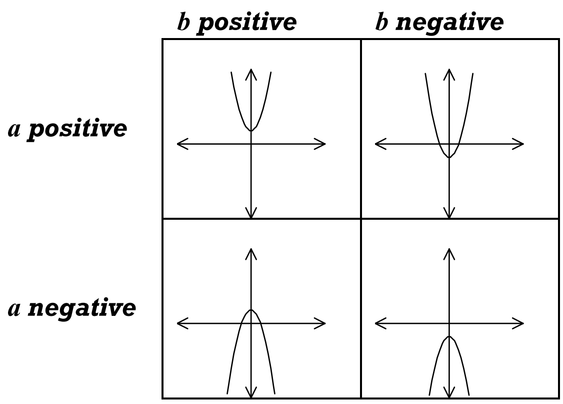
$y = ax^3 + b$

$y = \dfrac{a}{x+b}$

$y = x^a$, $a > 3$

$y = a^x$

Answers:
e.g., for $y = ax^2 + b$,

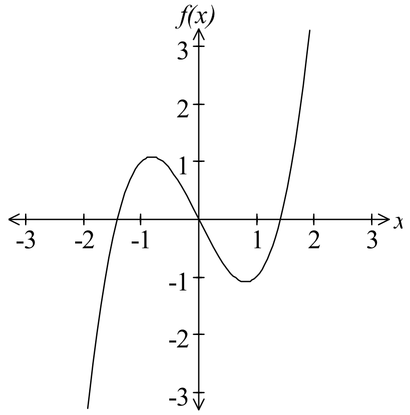


1.24.4 People graphs. Everyone stand up.
I want you to be $y = 0$ (arms out horizontally).
Now be $y = x$, $y = -x$ (aeroplanes!), $y = x^2$, $y = -x^2$, $y = x^3$, etc.

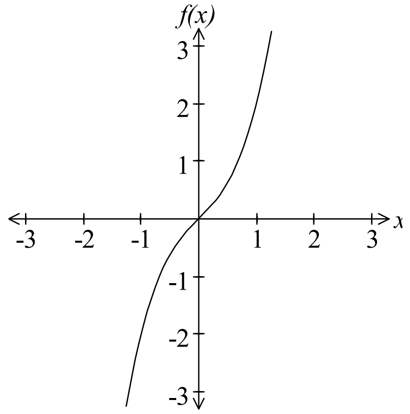
An interesting way of reviewing knowledge of the shapes of graphs!

To do $y = \sin x$ requires team-work!

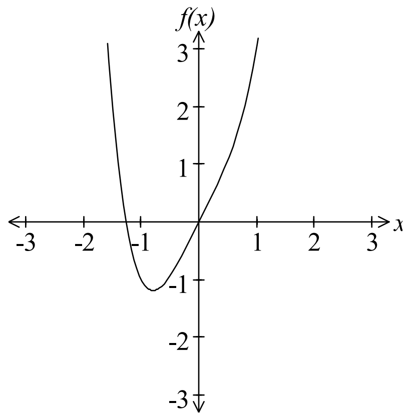
$f(x) = x^3 - 2x$



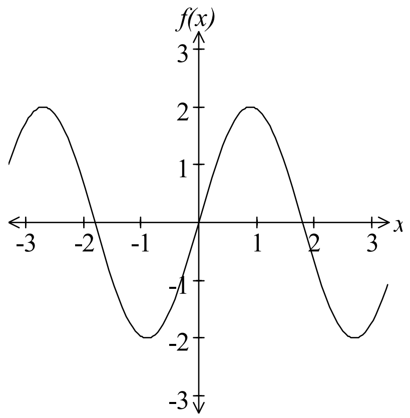
$f(x) = x^3 + x$



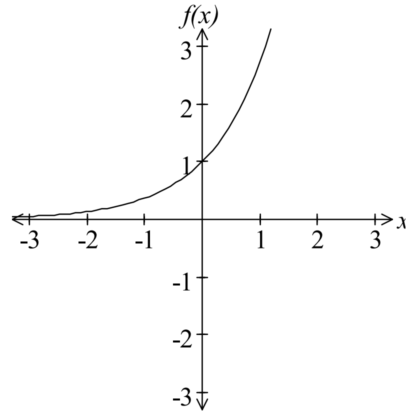
$f(x) = x^4 + 2x$



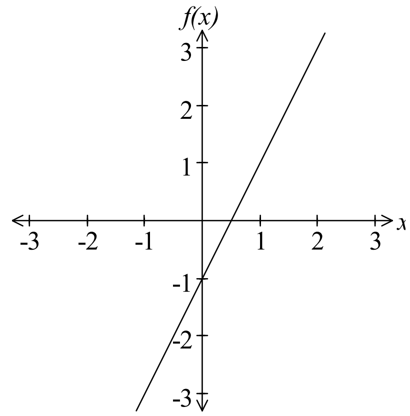
$f(x) = 2\sin 100x$



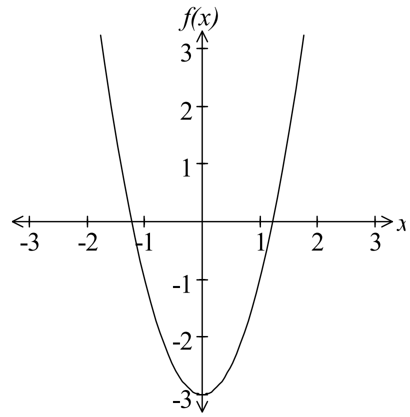
$f(x) = e^x$



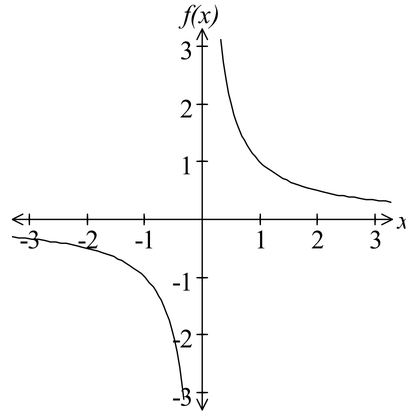
$f(x) = 2x - 1$



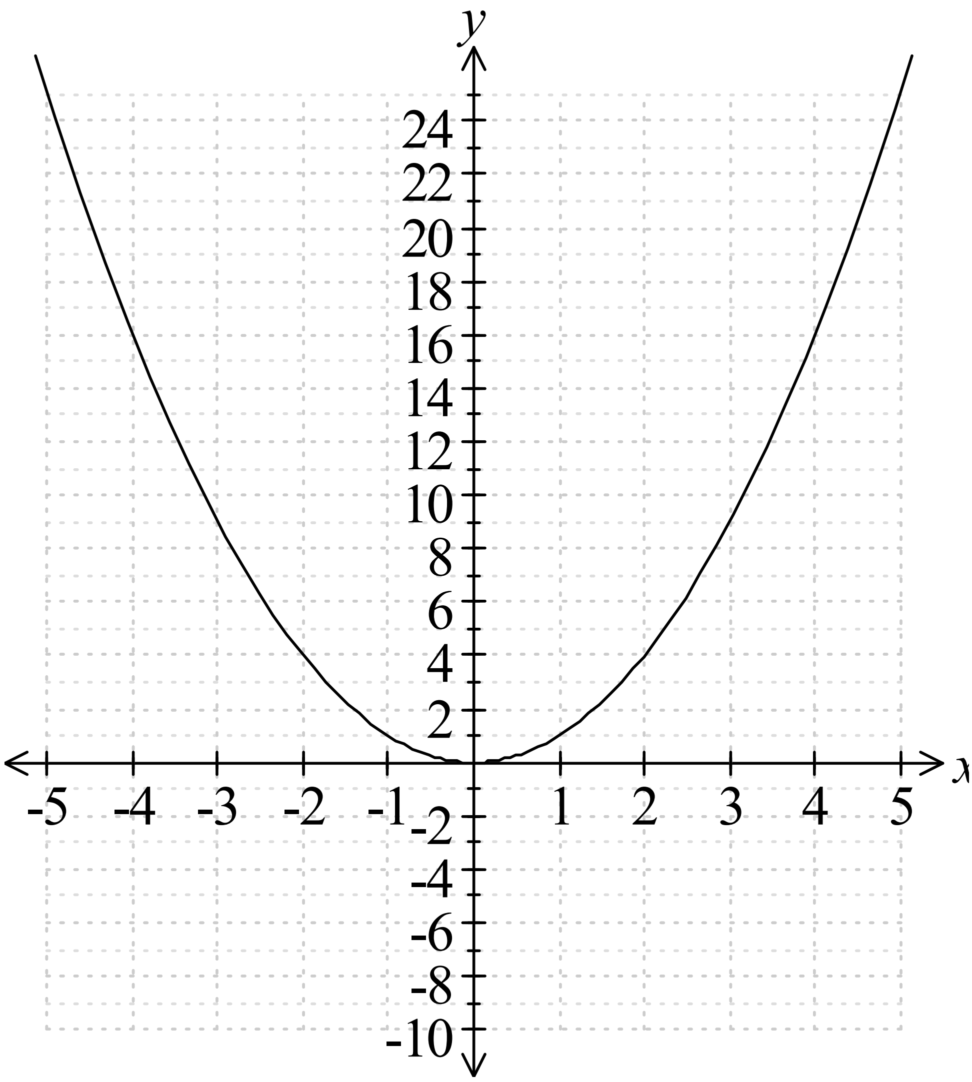
$f(x) = 2x^2 - 3$



$f(x) = \dfrac{1}{x}$



Finding the gradient function for $y = x^2$



Remember that the gradient of a straight line between two points A (x_A, y_A) and B (x_B, y_B) is $\frac{\Delta y}{\Delta x} = \frac{y_B - y_A}{x_B - x_A}$.

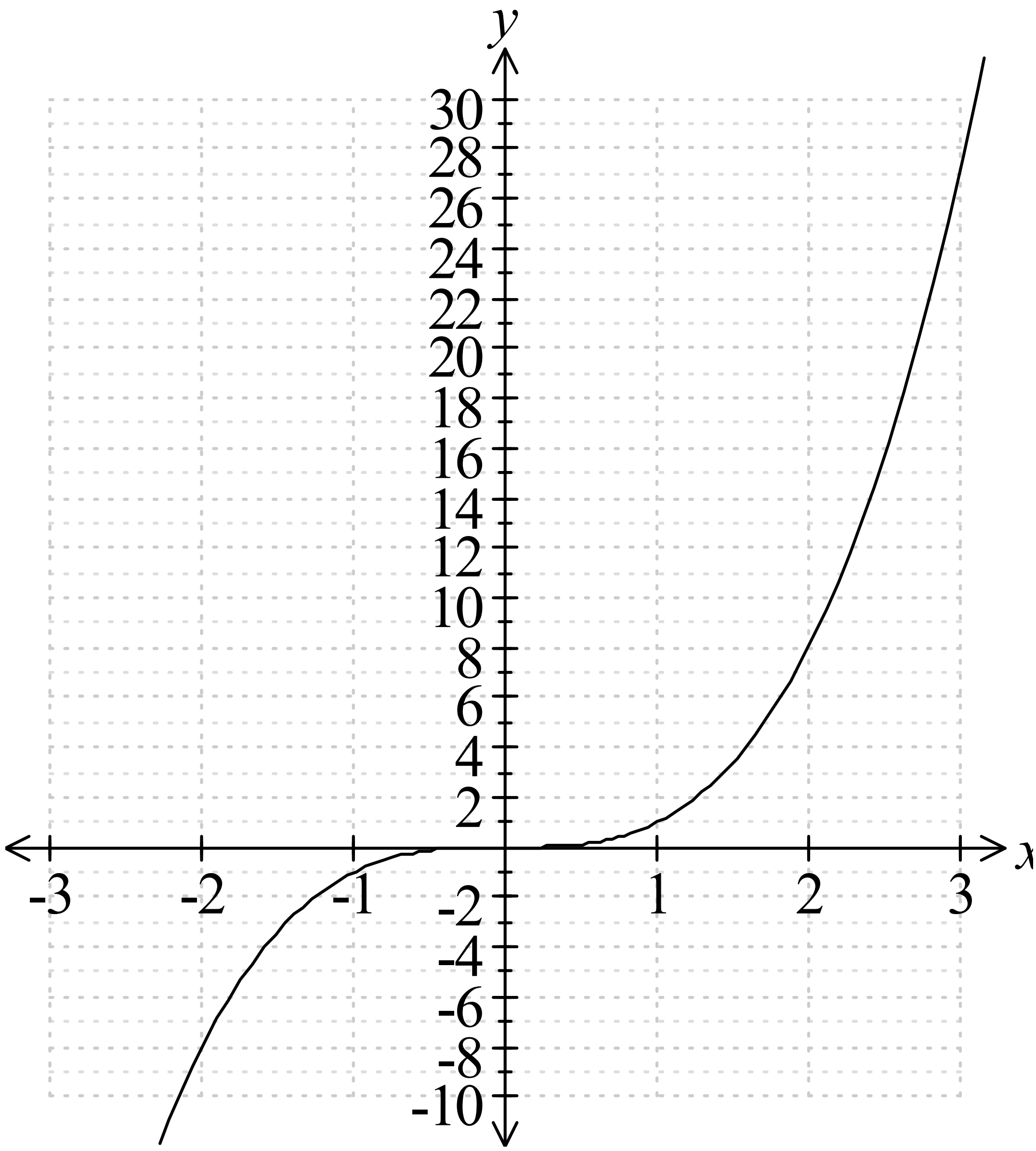
Fill in the table on the next page.
Then plot the *gradient function* on the graph above.
What is the equation of the gradient function graph?

Can you explain your results?

Finding the gradient function for $y = x^2$

point	nearby point	gradient	best gradient
(−5, 25)	(−4.9, 24.01)	−9.9	−10
	(−4.99,)		
	(−4.999,)		
(−4, 16)	(−3.9,)		
	(−3.99,)		
	(,)		
(−3,)	(,)		
	(,)		
	(,)		
(−2,)	(,)		
	(,)		
	(,)		
(−1,)	(,)		
	(,)		
	(,)		
(0, 0)	(0.1, 0.01)		
	(0.01,)		
	(,)		
(1,)	(1.1,)		
	(1.01,)		
	(,)		
(2,)	(2.1,)		
	(,)		
	(,)		
(3,)	(,)		
	(,)		
	(,)		
(4,)	(,)		
	(,)		
	(,)		
(5,)	(,)		
	(,)		
	(,)		

Finding the gradient function for $y = x^3$



Fill in the table on the next page.
Then plot the *gradient function* on the graph above.
What is the equation of the gradient function graph?

Can you explain your results?

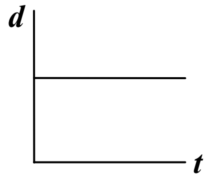
Finding the gradient function for $y = x^3$

point	nearby point	gradient	best gradient
(−3, −27)	(−2.9, −24.39)	26.11	27
	(−2.99,)		
	(−2.999,)		
(−2,)	(−1.9,)		
	(,)		
	(,)		
(−1,)	(,)		
	(,)		
	(,)		
(0,)	(0.1,)		
	(,)		
	(,)		
(1,)	(,)		
	(,)		
	(,)		
(2,)	(,)		
	(,)		
	(,)		
(3,)	(,)		
	(,)		
	(,)		

1.25 Real-Life Graphs

- A very valuable topic for understanding what a graph is/does/means. Possible contexts include distance-time (stones, cars, trains, etc.) and volume-time (water in pipes, etc.).
- It's worth trying to distinguish between *dependent* and *independent* variables. Which kind one is depends on the context. Sometimes it's an arbitrary choice. It's really a cause-and-effect relationship – the independent variable "causes" a change in the dependent one. If I go down the motorway at a constant 70 mph, how far I get (distance is the dependent variable) depends on time (the independent variable), but if I have to travel a fixed distance (say to London) then how long it takes me (time is the *dependent* variable this time) depends on my average speed (the independent variable).

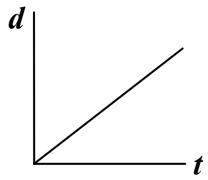
1.25.1 You could start with distance-time graphs. "This is me in my car. What's happening?"

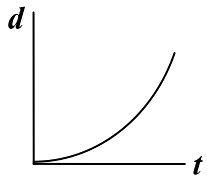


Answer:
"Run out of petrol", etc.

Expect "going along a flat road" and similar mistakes. Key questions are "how far have I got at this point?"; "how far have I got 1 minute later?". Put some numbers on the axes.

What's the difference between these two?

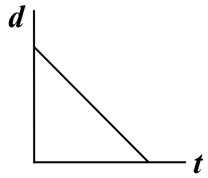




Steady speed versus accelerating.

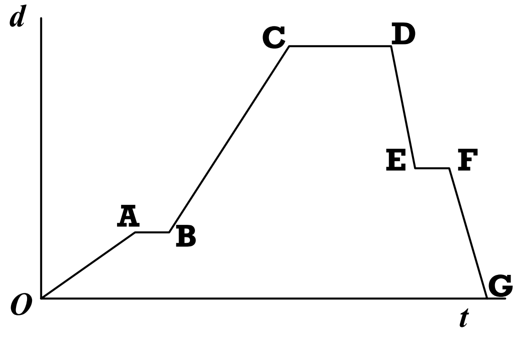
What would decelerating look like?
Answer: the curve would be reflected in the straight line; still going through the origin and distance increasing, but the slope getting shallower and shallower.

What's going on here?



Steady speed in the opposite direction.
Not necessarily downhill!

Draw on the board a record of a journey in my car (e.g., below). Pupils invent a story to fit it. They can put letters in brackets as they write to indicate where I am along the graph at that point.



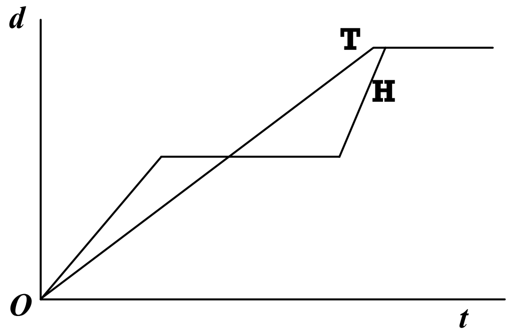
Stories often involve getting stopped by the police for speeding, so indicate in advance if you don't want this kind of humour!

Questions to sharpen up stories:
"When was I going fastest?" "Does that fit your story?" "Did I stop at the same place or a different place on the way back?" "If A to B was a traffic lights stop could I really have done my weekly shopping between C and D?!"
(You could allow pupils to modify their graph rather than re-write their whole story!)

Extension: Add scales to your drawing and work out the speeds at the different points.
Draw a speed-time or a velocity-time graph for the journey.

1.25.2 Draw a distance-time graph for the story of the hare and the tortoise. (You could ask one pupil to remind the rest of the story.)

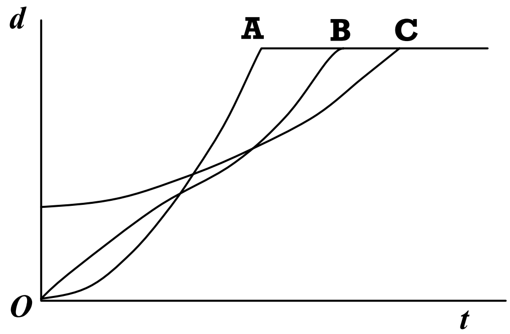
(The line should be steeper for the second leg than for the first leg of hare's journey. Both lines must be considerably steeper than the tortoise's line. Make sure the tortoise wins!)



1.25.3 Draw a distance-time graph for your journey to school today. Put approximate values on the axes and annotate it to explain what's going on. You could use vertical lines to separate bus-walking-train, etc. regions.

If the journey to school makes a rather dull graph, pupils could choose a different journey – to a friend's house, on holiday, etc.

1.25.4 Draw on the board a graph of 3 people running a race (e.g., below). Pupils write a race commentary. Who is in the lead at the start? Who overtakes who in what order? Was it a close finish? Did people run at steady speeds or did they all gradually get quicker? Did any of them seem to run out of steam?
e.g.,



C runs slowest – perhaps that's why they gave him a head start. Anyway, he speeded up but still came last.
A made a poor start but picked up speed and won.
B started fastest but seemed to "take it easy", and A speeded up and overtook him as he slowed down. Once he saw that A was going to win he increased his speed but it was too late and he only came in 2nd.

Pupils might make these details into a more realistic "commentary"-style.

1.25.5 Draw a graph of a *Skoda* accelerating away from a standstill at traffic lights. On the same graph draw a *Ferrari*.

Could discuss "0 to 60" values as a measure of acceleration.

1.25.6 Draw a distance-time graph for a dog running round in a circle (chasing its tail).

What if it gets faster and faster?

What about a cat chasing a dog in a circle?

Answer: to draw a normal distance-time graph, you need to consider just 1 direction of motion. Therefore, at a steady speed you get a sine wave. Motion at an increasing speed gives a sine wave of increasing frequency (decreasing time period). The cat's motion would also be sinusoidal at a similar frequency, though out of phase.

1.25.7 Alex runs round and round a race track taking 40 seconds to complete each circuit. Beth runs the opposite way round the track and meets Alex every 15 seconds. How long does it take Beth to complete each circuit of the track? Assume that they both run at constant speed and never get tired.

If Clare runs in the same direction as Alex and passes him every 60 seconds, how often do Clare and Beth pass each other?

Answers: Could sketch distance-time graphs and use similar triangles.

Alternatively, Alex's speed = $\frac{1}{40}$ *circuit/sec.*
If Beth's speed = b*, then* $\frac{1}{40}+b=\frac{1}{15}$ *so* $b=\frac{1}{15}-\frac{1}{40}=\frac{1}{24}$*, so Beth takes 24 seconds to complete a circuit. If Clare's speed =* c*, then* $c-\frac{1}{40}=\frac{1}{60}$ *so* $c=\frac{1}{40}+\frac{1}{60}=\frac{1}{24}$*, so Clare is running at the same speed as Beth. So they meet each other twice during each circuit, so they pass every 12 seconds.*

1.25.8 Graphs of height, diameter, etc. versus volume or time as liquids are poured at a steady rate into oddly-shaped containers.
You can actually do it reasonably well (qualitatively) with two jugs of water and some common containers (measuring cylinders, conical flask, beaker, test-tube from science dept.).

It may be useful to mention $\textbf{rate} = \dfrac{\textbf{how much}}{\textbf{how long}}$; *e.g., litres/min or mph or £ per kg, etc.*

Teacher demonstration is the safest!

1.25.9 A caterpillar crawls at a speed of 40 cm per hour towards a bed of cabbages. After eating his fill, he crawls back (more slowly) at only 10 cm per hour.
What was the caterpillar's average speed over the whole journey?
The answer *isn't* 25 cm/hour.

Answer: 16 cm per hour.
He spends four times as long returning as he took getting there (since he travels at only a quarter of the speed), so the mean speed is $\dfrac{1\times 40 + 4\times 10}{5} = \dfrac{80}{5} = 16$ *cm/hour.*
(You could draw a distance-time graph to visualise this.)

1.25.10 A train leaves London at 9.00 am heading for Edinburgh and travelling at 100 mph. At the same time a train leaves Edinburgh travelling at 120 mph towards London. When they pass each other, which train is closer to London?

Answer: They must be the same distance from London when they pass, obviously!

1.25.11 A train leaves Birmingham New Street Station and accelerates at a steady rate up to a speed of 100 mph. It maintains this speed for 30 mins before decelerating at a steady rate to a stop. If the whole journey takes 35 mins, how far has the train travelled?

Answer:
The easiest way of solving this is to draw a speed-time graph and calculate the area underneath it.
The graph is a trapezium of "height" 100 mph. The "top length" is 0.5 hours and the "bottom length" is the total journey time $\frac{35}{60}$ *hours.*
Therefore, the total distance = $\frac{1}{2}(a+b)h$
$= \frac{1}{2}(\frac{1}{2} + \frac{35}{60})100 = 54.2$ *miles.*

1.25.12 Pupils may have come across the *suvat* equations in Science, but may not realise that they relate to velocity-time graphs (see sheet).

These equations are valid only for constant (uniform) acceleration.

1.25.13 Zeno's Paradox of the Tortoise and Achilles (about 500 BC). The idea is that a tortoise challenges a man called Achilles to a race. Although the tortoise will obviously run slower than Achilles, the tortoise claims that he can still win so long as he gets a bit of a head start. Let's imagine that the race is 100 m and that Achilles can run at 10 m/s, whereas the tortoise can only do 1 m/s. Let's say that Achilles gives him a 10 m start. What will happen?

Here is the tortoise's (wrong) argument: One second after the race starts, Achilles will have run 10 m and is at the spot where the tortoise started, but he doesn't catch up with the tortoise yet because the tortoise has moved forward 1 m during that second. It takes Achilles another 0.1 second to catch up with where the tortoise is now, but by then the tortoise has moved on a further 10 cm. No matter how many times Achilles tries to catch him up, he'll always be a tiny bit behind – he'll never be able to overtake him.

This can make an interesting class or group discussion.

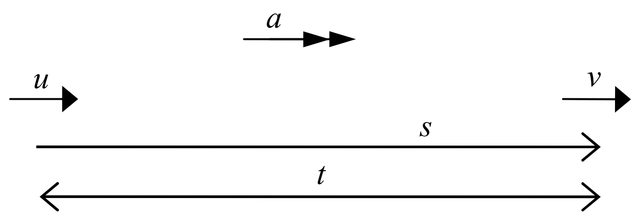
A paradox *is something that sounds right but doesn't seem to make sense.*

These aren't realistic values; they're just nice numbers to make the paradox easier to grasp.

Commonsense says that Achilles will take 10 seconds to get to the finish, whereas the tortoise will take 90 seconds (he has to run only 90 metres), so Achilles will still win by 80 seconds. This is in fact correct.

The tortoise is wrong because after 1.1111… seconds Achilles will catch him up and then overtake him – and after 10 seconds he'll have won the race. The mistake is to think that because the time 1.111… (recurring) seconds has infinitely many digits it is an infinite amount of time; it isn't, it's in between 1.1 and 1.2 seconds. ($10t = 10 + t \Rightarrow t = \frac{10}{9}$ *when they meet.)*

Displacement, Velocity, Acceleration



A particle initially has a velocity u and moves a displacement s under an acceleration a so that it ends up with a final velocity v at a time t later on.

There are five so-called *suvat* equations:

Formulas for straight-line motion under constant acceleration a

$v = u + at$
(no displacement s)

$s = ut + \frac{1}{2}at^2$
(no final velocity v)

$s = vt - \frac{1}{2}at^2$
(no initial velocity u)

$s = \frac{1}{2}(u+v)t$
(no acceleration a)

$v^2 = u^2 + 2as$
(no time t)

Different Units for Speed

mph

0 5 10 15 20 25 30 35 40 45 50 55 60 65 70 75 80 85 90 95

kph

0 10 20 30 40 50 60 70 80 90 100 110 120 130 140 150

m/s

0 2 4 6 8 10 12 14 16 18 20 22 24 26 28 30 32 34 36 38 40 42

$$\textbf{knot} \underset{\times 1.15}{\overset{\div 1.15}{\rightleftarrows}} \textbf{mph} \underset{\div 1.6}{\overset{\times 1.6}{\rightleftarrows}} \textbf{kph} \underset{\div \frac{5}{18}}{\overset{\times \frac{5}{18}}{\rightleftarrows}} \textbf{m/s}$$

1.26 Inequalities

- This is a topic which builds heavily on several others: negative numbers and number-lines; solving linear (or other) equations; plotting and interpreting $y = mx + c$ and other graphs. It may be worth reviewing some of these at the outset, depending on what you're aiming to cover.
- It may be quite a jump from saying that if $2x + 3 = 25$ then $x = 11$ to saying that if $2x + 3 > 25$ then $x > 11$, and this can be especially hard to see if negative numbers are involved. It might be helpful to go back to the balancing ideas of solving equations: here we have some unbalanced scales (the heavy left side is down and the right side is up), and if we add the same amount to both sides the imbalance will remain just as it was (and to the same degree).
- One way to approach this topic is to put a hardish inequality on the board (e.g., $12x - 31 > 60$) and ask for any number that will work; e.g., $x = 100$ or $x = 1000000$, and then start looking for the *smallest* number that will work ($\frac{91}{12} = 7.583...$). This can be by trial and improvement initially.
- When shading several graphical inequalities on xy axes, it is often simpler to shade *out* the regions that you *don't* want so that the un-shaded portion left at the end represents the required points. The only problem with this is that the convention of using solid lines for < and > and dashed lines for ≤ and ≥ becomes ambiguous. We have to assume that those conventions apply to the *required* region.

1.26.1 Algebra is not just about equations (when the two sides of the scales balance) but also about when one side is heavier than the other. There are four inequality signs: $<$, $>$, $\le$ and $\ge$; the first two just more precise ways of saying $\ne$ (not equal to).

You can do "true and false" statements to review these definitions.

A common pupil question is, e.g., "Does $x > 17$ include 17?" which you could answer by asking, "Is 17 more than 17?" No.

These can be quite hard, especially if they include negative or decimal numbers.
"Stand up if true, sit down if false" can be fun.

1.26.2 Number-lines help with seeing that $x < 3$ *and* $x < -2$ (simultaneously true) means that $x < -2$, whereas $x > 3$ *and* $x < -2$ describe no possible values of x.
Also that $x < 3$ *and* $x > -2$ means "between" and can be written as $-2 < x < 3$.

When illustrating inequalities on a number-line, the usual convention is to use a coloured-in *dot to mark the end of an interval if the end is* included*, and an* open *dot if the end-point is* not included.

1.26.3 I'm thinking of an integer (or maybe more than one integer).

1. The number I'm thinking of is less than 40.
2. If I double the number, the answer is more than 55.
3. If I take the number away from 100, I get more than 60.
4. If I multiply the number by 3 and add 4 I get more than 113.

What could my number be?

How many possibilities are there?
Pupils could investigate with a calculator.

Answers: 37, 38, 39

Note that condition 3 duplicates condition 1 (it is redundant).

1.26.4 Are these always true (can you prove it?), sometimes true (tell me when) or never true (tell me why)?

1. $a + b < a - b$
2. $ab < a + b$
3. $ab < \frac{a}{b}$

Answers: ("iff" means "if and only if")

1. *true iff $b < 0$;*
2. *true iff a and b are both < 2;*
3. *true iff $b < -1$ and $a > 0$;*
   or $-1 < b < 0$ and $a < 0$;
   or $0 < b < 1$ and $a > 0$;
   or $b > 1$ and $a < 0$.

1.26.5 True or false?
If $0 < a < 1$, $b > 1$, $-1 < c < 0$ and $d < -1$,

1. $ab < b$
2. $\frac{b}{a} < b$
3. $\frac{a}{b} < a$
4. $ab < a$
5. $a + b > b$
6. $a + c > a$
7. $a + c > c$
8. $ac > a$
9. $bc < b$
10. $cd > c$

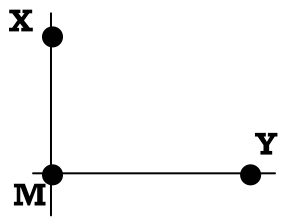
Answers: Pupils can try these with numbers or try to argue through them logically. Not easy.
1. T
2. F
3. T
4. F
5. T
6. F
7. T
8. F
9. T
10. T

1.26.6 A lift has a mass of 820 kg. If the cable can safely tolerate a maximum load of 1400 kg, how many people (of average mass 70 kg) can it hold?

Answer: 8 people (always round down where safety is concerned!)
$70x + 820 \leq 1400$ *gives* $x = 8.29...$

1.26.7 Imagine this situation:
Eighty pupils are lined up in a rectangular array that is 8 rows by 10 columns. I go along every column noting the shortest person in that column. I then pick out the *tallest* of these 10 pupils and give him/her a coloured hat marked with an X.
Next I go along every *row* noting the *tallest* person in that row. Then I pick out the *shortest* of those 8 pupils and give him/her a coloured hat marked with a Y.
Which person is taller, person X or person Y, or can't we be sure?

Answer:
Imagine X and Y in the grid and also pupil M, who is in the same column *as X and same* row *as Y. It doesn't matter how many rows and columns there are in between.*



X is the shortest person in the column, so M must be taller than X ($h_M > h_X$ *).*
(h_M *stands for the height of person M, etc.)*
Y is the tallest person in the row, so Y must be taller than M ($h_Y > h_M$ *).*
Therefore Y is taller than X ($h_Y > h_M > h_X$ *).*

Could person X and person Y be the same person?

Yes, it could happen if there are bigger people in the rest of the column and smaller people in the rest of the row.

1.26.8 Quadratic Inequalities.
The easiest way to solve these is to factorise (or use the equation) so as to write the inequality as something like $(x-a)(x-b) \geq 0$.
Then sketch the graph $y = (x-a)(x-b)$ to see where it is ≥ 0; e.g., the solution of $(x-2)(x+3) \geq 0$ is $x \leq -3$ or $x \geq 2$ because for these values of x the graph is on or above the x-axis.

It is possible to solve these without drawing graphs, but you have to think very logically. e.g., if the product of $(x-2)$ *and* $(x+3)$ *is* ≥ 0*, it means that both must be* ≥ 0 *or both must be* ≤ 0*, so the possibilities are*
(i) $x - 2 \geq 0$ ***and*** $x + 3 \geq 0$*;* ***OR***
(ii) $x - 2 \leq 0$ ***and*** $x + 3 \leq 0$.
Now (i) $\Rightarrow x > 2$ *and (ii)* $\Rightarrow x < -3$*, so these are the solutions.*
Where the given inequality is < 0 *or* ≤ 0 *instead, the two brackets must be of* opposite *sign (or zero) and the same method will work.*

1.26.9 Linear Programming.
This can be an interesting application of graphical inequalities.

Key Stage 3 Strategy – Key Objectives Index

These are the key objectives from the *Key Stage 3 Strategy* (DfES, 2001) with references to sections of relevant material from all three volumes.

Year 7

Simplify fractions by cancelling all common factors; identify equivalent fractions.	1.6
Recognise the equivalence of percentages, fractions and decimals.	1.11
Extend mental methods of calculation to include decimals, fractions and percentages.	1.2-11, 3.6
Multiply and divide three-digit by two-digit integers; extend to multiplying and dividing decimals with one or two places by single-digit integers.	1.5, 3.6
Break a complex calculation into simpler steps, choosing and using appropriate and efficient operations and methods.	various
Check a result by considering whether it is of the right order of magnitude.	1.15, 2.15-16
Use letter symbols to represent unknown numbers or variables.	1.19-22, 1.26
Know and use the order of operations and understand that algebraic operations follow the same conventions and order as arithmetic operations.	1.12, 1.20
Plot the graphs of simple linear functions.	1.23
Identify parallel and perpendicular lines; know the sum of angles at a point, on a straight line and in a triangle.	2.4-5
Convert one metric unit to another (e.g., grams to kilograms); read and interpret scales on a range of measuring instruments.	2.15, 1.2
Compare two simple distributions using the range and one of the mode, median or mean.	3.3
Understand and use the probability scale from 0 to 1; find and justify probabilities based on equally likely outcomes in simple contexts.	3.5
Solve word problems and investigate in a range of contexts, explaining and justifying methods and conclusions.	various

Year 8

Add, subtract, multiply and divide integers.	1.3, 3.6
Use the equivalence of fractions, decimals and percentages to compare proportions; calculate percentages and find the outcome of a given percentage increase or decrease.	1.9-11
Divide a quantity into two or more parts in a given ratio; use the unitary method to solve simple word problems involving ratio and direct proportion.	1.10
Use standard column procedures for multiplication and division of integers and decimals, including by decimals such as 0.6 or 0.06; understand where to position the decimal point by considering equivalent calculations.	1.2-3, 1.5, 3.6
Simplify or transform linear expressions by collecting like terms; multiply a single term over a bracket.	1.20
Substitute integers into simple formulas.	1.20
Plot the graphs of linear functions, where y is given explicitly in terms of x; recognise that equations of the form $y = mx + c$ correspond to straight-line graphs.	1.23
Identify alternate and corresponding angles; understand a proof that the sum of the angles of a triangle is 180° and of a quadrilateral is 360°.	2.4
Enlarge 2-d shapes, given a centre of enlargement and a positive whole-number scale factor.	2.12-13
Use straight edge and compasses to do standard constructions.	2.8
Deduce and use formulas for the area of a triangle and parallelogram, and the volume of a cuboid; calculate volumes and surface areas of cuboids.	2.2, 2.9-10
Construct, on paper and using ICT, a range of graphs and charts; identify which are most useful in the context of a problem.	1.23-25, 3.2, 3.7
Find and record all possible mutually exclusive outcomes for single events and two successive events in a systematic way.	1.5
Identify the necessary information to solve a problem; represent problems and interpret solutions in algebraic, geometric or graphical form.	various
Use logical argument to establish the truth of a statement.	various

Year 9

Add, subtract, multiply and divide fractions.	1.7-8
Use proportional reasoning to solve a problem, choosing the correct numbers to take as 100% or as a whole.	1.9-10
Make and justify estimates and approximations of calculations.	1.4, 2.15-16
Construct and solve linear equations with integer co-efficients, using an appropriate method.	1.18, 1.20
Generate terms of a sequence using term-to-term and position-to-term definitions of the sequence, on paper and using ICT; write an expression to describe the n^{th} term of an arithmetic sequence.	1.19, 3.7
Given values for m and c, find the gradient of lines given by equations of the form $y = mx + c$.	1.23
Construct functions arising from real-life problems and plot their corresponding graphs; interpret graphs arising from real situations.	1.24-25, 3.2
Solve geometrical problems using properties of angles, of parallel and intersecting lines, and of triangles and other polygons.	2.1, 2.4-5
Know that translations, rotations and reflections preserve length and angle and map objects onto congruent images.	2.12-13
Know and use the formulas for the circumference and area of a circle.	2.3
Design a survey or experiment to capture the necessary data from one or more sources; determine the sample size and degree of accuracy needed; design, trial and if necessary refine data collection sheets.	3.1
Communicate interpretations and results of a statistical enquiry using selected tables, graphs and diagrams in support.	3.2-3
Know that the sum of probabilities of all mutually exclusive outcomes is 1 and use this when solving problems.	3.5
Solve substantial problems by breaking them into simpler tasks, using a range of efficient techniques, methods and resources, including ICT; give solutions to an appropriate degree of accuracy.	1.4, 3.7, various
Present a concise, reasoned argument, using symbols, diagrams, graphs and related explanatory text.	various

Year 9 (extension)

Know and use the index laws for multiplication and division of positive integer powers.	1.14
Understand and use proportionality and calculate the result of any proportional change using multiplicative methods.	1.9-10
Square a linear expression and expand the product of two linear expressions of the form $x \pm n$; establish identities.	1.20-21
Solve a pair of simultaneous linear equations by eliminating one variable; link a graphical representation of an equation or a pair of equations to the algebraic solution.	1.22
Change the subject of a formula.	1.20
Know that if two 2-d shapes are similar, corresponding angles are equal and corresponding sides are in the same ratio.	2.12
Understand and apply Pythagoras' theorem.	2.7
Know from experience of constructing them that triangles given SSS, SAS, ASA or RHS are unique, but that triangles given SSA or AAA are not; apply these conditions to establish the congruence of triangles.	2.12
Use measures of speed and other compound measures to solve problems.	2.16
Identify possible sources of bias in a statistical enquiry and plan how to minimise it.	3.1
Examine critically the results of a statistical enquiry and justify choice of statistical representation in written presentations.	3.1-3
Generate fuller solutions to mathematical problems.	various
Recognise limitations on the accuracy of data and measurements.	1.4