instant maths ideas

FOR KEY STAGE 3 TEACHERS

shape and space

nelson thornes

Volume 2

Shape and Space

Colin Foster

Introduction

Teachers are busy people, so I'll be brief.
Let me tell you what this book *isn't*.

- It *isn't* a book you have to make time to read; it's a book that will *save* you time. Take it into the classroom and use ideas from it straight away. Anything requiring preparation or equipment (e.g., photocopies, scissors, an overhead projector, etc.) begins with the word "**NEED**" in bold followed by the details.

- It *isn't* a scheme of work, and it isn't even arranged by age or pupil "level". Many of the ideas can be used equally well with pupils at different ages and stages. Instead the items are simply arranged by topic. (There is, however, an index at the back linking the "key objectives" from the *Key Stage 3 Framework* to the sections in these three volumes.) The three volumes cover ***Number and Algebra*** (1), ***Shape and Space*** (2) and ***Probability, Statistics, Numeracy and ICT*** (3).

- It *isn't* a book of exercises or worksheets. Although you're welcome to photocopy anything you wish, photocopying is expensive and very little here needs to be photocopied for pupils. Most of the material is intended to be presented by the teacher orally or on the board. Answers and comments are given on the right side of most of the pages or sometimes on separate pages as explained.

This is a book to make notes in. Cross out anything you don't like or would never use. Add in your own ideas or references to other resources. Put "8R" (for example) next to anything you use with that class if you want to remember that you've used it with them.

Some of the material in this book will be familiar to many teachers, and I'd like to thank everyone whose ideas I've included. I'm particularly grateful to those people who have discussed some of these ideas with me; especially Keith Proffitt, Paul Andrews, John Cooper and Simon Wadsley. Special thanks go to Graham Foster for expert computer behaviour management!

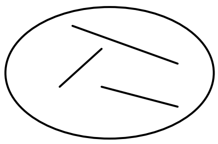
Colin Foster
July 2003

Contents

Volume 2 – Shape and Space

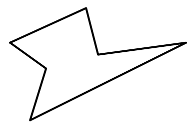
2.1 Polygons

- A topic containing lots of definitions. One way to make this interesting is for pupils to look for "hard cases" that get around other people's definitions. Or the teacher can do that at the board as the pupils attempt to define key concepts;



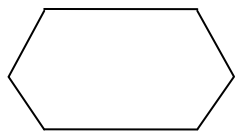
e.g., "A polygon is a shape containing straight lines", so the teacher draws something like this (left), and the pupils have to think of a better definition. "I'm going to be awkward – try and come up with a definition I won't be able to get around." You might eventually end up with something like "a flat closed shape made up entirely of straight sides", or better.

- It's worth emphasising that odd-looking, non-standard polygons (e.g., see right), are still polygons (it's even a hexagon), and that there's nothing "wrong" with them.



- "Regular" means that all the sides have the same length *and* all the angles are equal (or "all the vertices look the same" if "angle" is not yet a clear concept). It's helpful to see that both of these conditions must hold by imagining irregular hexagons like those below.

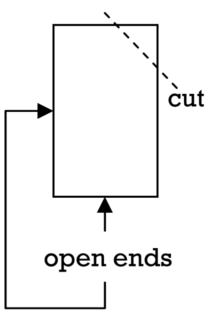
all sides the same length, but angles different sizes



all angles the same size, but sides different lengths

  So the only *regular* quadrilateral, for instance, is the square (e.g., you can't have a "regular trapezium", etc.).
- Material involving angles in polygons is in section 2.4.

2.1.1 **NEED** newspaper, scissors and practice!
Fold a whole sheet of newspaper in half and then in half again. The teacher then cuts a shape out of the corner which corresponds to the centre of the original sheet.



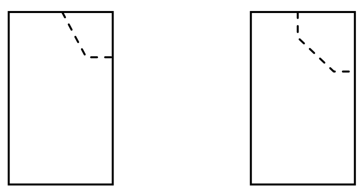
"When I open it out, what will we see?"
A hole in the middle.
"What shape will the hole be?"

Then open and see.
"What shape is it?"

Pupils can use scrap paper or newspaper to try to make particular shapes.
You could discuss reflection symmetry.

Answers:
Ask pupils to predict each time before you open out the newspaper.

1. *Cut a straight line at 45° to both edges to get a **square**. (Expect pupils to say "diamond" because of the orientation.)*
2. *A similar cut not at 45° makes a **rhombus**.*
3. *Cut lines at 120° in a 2:1 ratio as below (left) to make a **regular hexagon**.*
4. *Cut lines at 135° in a 1:2:1 ratio as below (right) to make a **regular octagon**.*



It's possible to make many other shapes by experimenting.

2.1.2 "I spy a polygon!". Looking for polygons in the classroom or around the school.
So long as they don't have to be regular, there should be lots. You can declare that "squares and rectangles are boring".

This can be a brief task at the beginning or end of a lesson.

You can offer challenges such as "Can anyone see a heptagon?"

2.1.3	Names of polygons. Make a table (see sheet). Discuss how pupils are going to remember the names. Where do you come across these shapes? • quadrilateral: is there a "quad"/court in school? • pentagon: "Pentagon" in US; • hexagon: they tessellate in bee-hives; • heptagon: 20p and 50p coins, although they're actually a little rounded at the corners; • octagon: an octopus has 8 tentacles; • decagon : "decimal", "decimetre", etc.	*1- and 2-sided polygons don't exist; the only special names for* regular *polygons are "equilateral triangle" (3) and "square" (4); otherwise we just say "regular" before the name.* *The US Pentagon was built in that shape with the idea that it would be quick to get from any part of the building to any other part.* *Names for polygons with lots of sides are interesting to some pupils, although we would probably say "46-gon", etc. (see sheet).*
2.1.4	Where is there a very large, very well-known triangle? Where exactly is it?	*Answer: (there may be other answers)* *The Bermuda Triangle, in which many planes and ships have gone missing over the years. Its vertices are at Bermuda, Miami (Florida) and San Juan (Puerto Rico).*
2.1.5	Which letter of the Greek alphabet looks like a triangle?	*Answer: Capital delta, the fourth letter of the Greek alphabet, is* Δ *(the lower case delta is* δ*), and is used in maths and science, as is the upside down version* ∇*.*
2.1.6	**NEED** square dotty paper, or photocopies of sets of 3 × 3 squares of dots (see sheets). If every vertex must lie on a dot, how many different triangles can you draw on a 3 × 3 square grid of dots? Count as the same any triangles which are just reflections, rotations or translations of each other?	*Answers:* *Equilateral triangles are impossible.* *See sheet for the others.* *You can do a similar task with quadrilaterals (see sheet).*
2.1.7	**NEED** acetate of quadrilaterals (see sheet). "What have all these shapes got in common?" (polygons, 4 sides, quadrilaterals) "Pick one and tell me what you would call it." "What makes it an X? What does a shape have to have to make it an X?" You can offer a challenge: "Who thinks they could say the name of every shape?"	*This can lead to seeing that all squares are rectangles, rhombuses and parallelograms, etc.* *This may be the time to introduce the notation for equal angles, equal sides and parallel sides.* *You can turn the acetate by quarter turns and even turn it over to change the appearance and positions of the shapes.*
2.1.8	Classifying Quadrilaterals (see sheet). This is more complicated than it may seem at first sight. You need very careful definitions.	*Construct a Venn Diagram or a Flow Diagram for classifying any quadrilateral; e.g., "Are all the sides equal? Y/N", etc.* *The table at the bottom of sheet (or one like it) can be drawn on the board and completed by pupils (individually or in groups).*
2.1.9	**NEED** photocopies, scissors and glue. Matching definitions (see sheet). Pupils could work in pairs or individually. Cut out the statements and the polygon names and match them up. Could stick them down in books if you want a permanent record.	*This fits nicely on a double page of a normal exercise book.*
2.1.10	Link polygons to co-ordinates (all positive or positive and negative), and kill two birds with one stone.	*Pupils can make up their own.* *("Plot these points and join them up – name the resulting polygon.")*

2.1.11 Choose a volunteer. They stand at the front of the room. Write the name of a polygon on a piece of paper and show it to the pupil. The pupil has to describe it without using the word you've shown them, without drawing anything on the board or waving arms around. When enough information has been given, the pupil chooses another to "guess", and if correct that pupil replaces the one at the front.

You may sometimes need to penalise guessing by deducting points for wrong guesses.

You can do this in teams or against the clock.

Obviously make sure that the class can't see the word through the paper! (Whispering the word to the pupil is likely to be too insecure!)

You need to decide whether you will allow things like "you tie string to it and it flies in the sky"! Really the aim is to be talking about the mathematical properties of the shapes!

Sometimes you may need to interfere because pupils guess correctly from poor explanations. "Did it have to be that?" "Have you got enough information yet to rule out every other possibility?"

2.1.12 **NEED** "Finding Quadrilaterals" sheet. Which kind of quadrilateral isn't there?

Pupils need to use the ABCD labelling convention (clockwise or anticlockwise, start anywhere, don't need to repeat the vertex you start at).

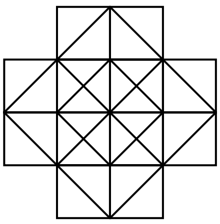
Can also look for different kinds of triangles and for polygons with more than 4 sides.

The drawing is accurate, so pupils can measure lines and angles.
A square is the only one missing.

All the sides of the shape have to be lines that are actually drawn in. If you draw in more lines then there are too many polygons to find.

Pupils can invent their own version, but may need advising not to make it too complicated!

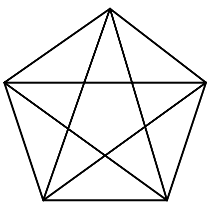
2.1.13 How many squares of any size can you find in this drawing?



Answer: 27
Be systematic:
- *side length 1, there are 12;*
- *side length 2, there are 5;*
- *side length* $\frac{1}{2}\sqrt{2}$ *, there are 4;*
- *side length* $\sqrt{2}$ *, there are 5;*
- *side length* $2\sqrt{2}$ *, there is 1.*

So the total is 27.

2.1.14 How many triangles of any size can you find in this drawing of a pentagram inside a pentagon?



Answer: 35
Be systematic again:
(edge means edge of the large pentagon)
- *small isosceles, there's 5;*
- *large isosceles, there's 5;*
- *acute-angled containing 1 edge, there's 5;*
- *obtuse-angled containing 1 edge, there's 10;*
- *obtuse-angled containing 2 edges, there's 5;*
- *obtuse-angled inside, there's 5.*

So the total is 35.

2.1.15 Imagination (see sheet).
Pupils may prefer to close their eyes when trying to visualise these.
The teacher can read them out slowly.

Answers: ("Clock Polygons")
1. an equilateral triangle; 2. a square; 3. a regular hexagon; 4. an isosceles triangle; 5. a rectangle; 6. a kite; 7. a scalene triangle; 8. (hard) a right-angled triangle (the angle in a semicircle is always 90˚); 9. a trapezium; 10. an irregular pentagon.

Answers: ("Shape Combinations")
1. *an obtuse-angled isosceles triangle OR a parallelogram;*
2. *a different parallelogram OR an arrowhead;*
3. *a rhombus;*
4. *a parallelogram OR a kite;*
5. *a triangle (if you do it to all 8 vertices, the solid you end up with is called a "truncated cube");*
6. *a hexagon (not necessarily a regular one);*
7. *7. a parallelogram OR a concave hexagon.*

2.1.16 Polygon People.
Use pencil and ruler to draw a "polygon person" made entirely out of polygons.
Underneath make a table of "body part" and "polygon name".

You can restrict this in some way (e.g., to just triangles). Can make nice display work. Polygon animals/aliens are obvious alternatives.

Concave polygons are still polygons.

2.1.17 Start with a square piece of paper. With one straight cut, what shapes can you make?

A good task for promoting "exhaustive thinking" (considering all the possibilities).

Answer: 2 rectangles (congruent or not); or 2 congruent right-angled isosceles triangles; or 1 right-angled triangle and 1 irregular pentagon containing 3 right-angles; or 1 right-angled triangle and 1 right-angled trapezium (depending on the angle of the cut and whether it goes through 0, 1 or 2 vertices).

What if you are allowed 2 straight cuts?

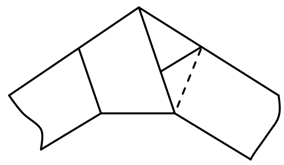
Lots of possibilities now. You can find them all by drawing the 4 possibilities above and considering all the positions of a second line: it could pass through 0, 1 or 2 vertices; if the first line went through a vertex the second one may or may not go through the same vertex; lines parallel and perpendicular to the first line may give different possibilities.

2.1.18 Make a poster of polygon vocabulary illustrating each word to make it easier to remember; e.g., making the double-l in "parallelogram" into a pair of parallel lines.

para||elogram

2.1.19 **NEED** scrap paper, scissors.
Making a Pentagon.
Cut out a thin strip of paper with the same width all the way along.
Tie a very loose knot and flatten it down.
What shape do you expect to get?

Answer: regular pentagon



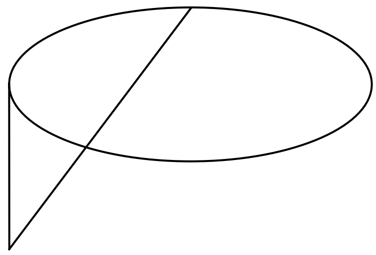
2.1.20 **NEED** Tangrams (bought or made).
You can buy plastic sets of pieces or make your own out of 1 cm × 1 cm A4 squared paper (or A4 card with the shapes photocopied onto it – see sheet).
Many different objects/pictures can be made, ranging from fairly easy to extremely difficult.

All the pieces must be used in each puzzle, and no overlapping is allowed.

It's worth examining the pieces carefully. This happens naturally if pupils make their own. The most common tangram set (see sheet) comes from cutting up a 4 × 4 square into 7 pieces. There are 2 pairs of congruent right-angled isosceles triangles, another right-angled isosceles triangle, a square and a parallelogram.
The only one worth turning over is the parallelogram (it's the only one without at least 1 line of symmetry).

2.1.21 Describing Designs.
In pairs, pupils turn their chairs so they are sitting back-to-back. One pupil draws a shape or combination of shapes (not too complicated) and the other has some rough paper. The first pupil has to describe orally the shape so that the second can accurately draw it without either pupil seeing the other's paper. The second person isn't allowed to speak. The final shape has to be in the same orientation and about the same size as the original. Team-work is the aim.

e.g., quite a difficult one would be



It's easier to say what to do *rather than what is* there*; e.g., "Put your pen at the centre of the paper and draw a line straight down for about 6 cm" rather than "there's a 6 cm straight line down the middle of the page".*

Polygon Names

number of sides	name	name if regular
1	-	-
2	-	-
3	triangle	equilateral triangle
4	quadrilateral	square
5	pentagon	-
6	hexagon	-
7	heptagon	-
8	octagon	-
9	nonagon (or enneagon)	-
10	decagon	-
11	undecagon (or hendecagon)	-
12	dodecagon	-

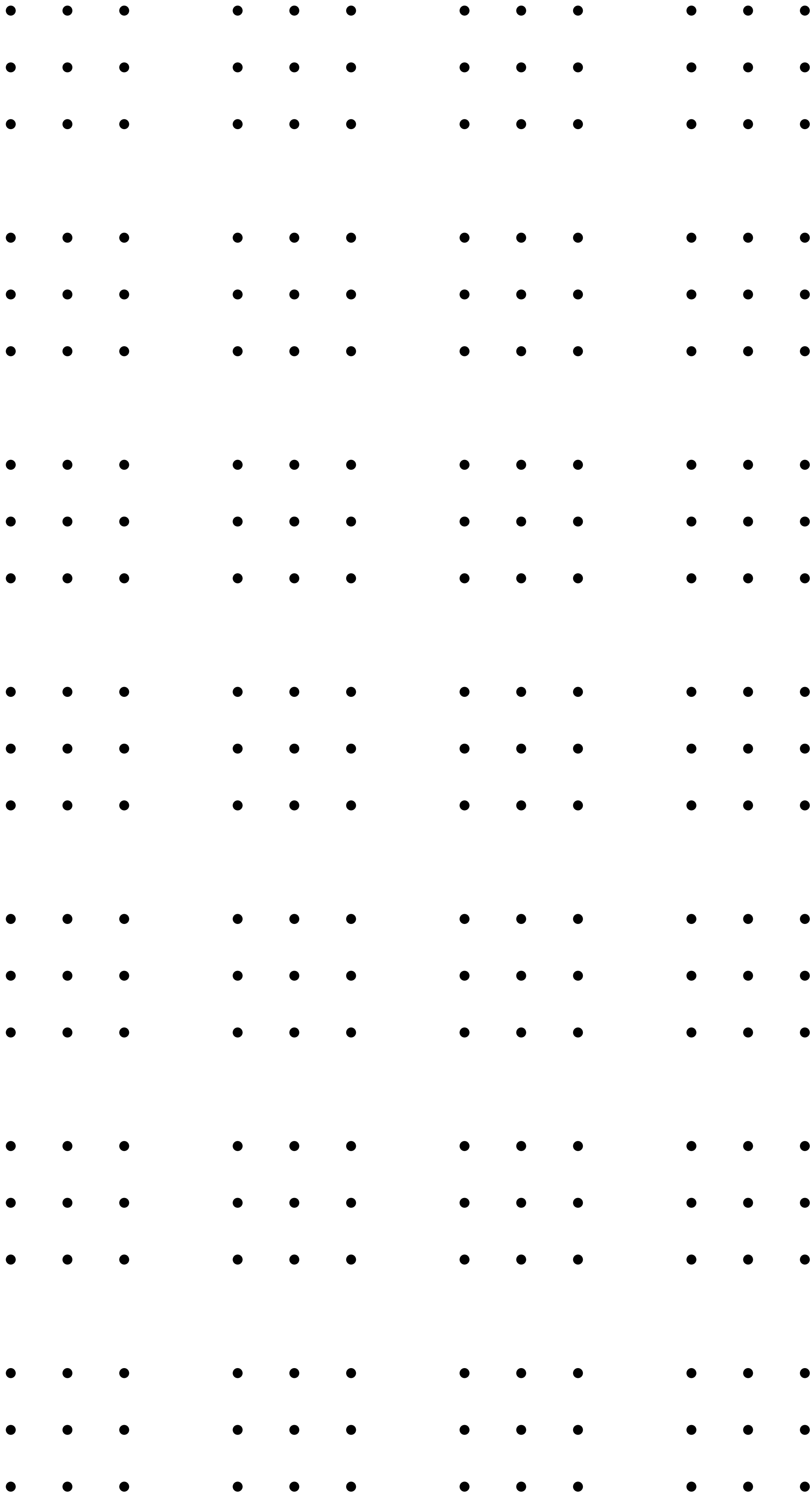
- *Equilateral triangle* (equal sides) and *equiangular triangle* (equal angles) both refer to a regular triangle, whereas a quadrilateral has to be *both* equilateral (rhombus) *and* equiangular (rectangle) to be a square.
- Although other quadrilaterals than squares are common ("regular" in the sense of ordinary), they are not *mathematically regular* because they don't have all their sides of equal length and all their angles the same size.
- A dodecagon has 12 sides; a dodecahedron is a 3-d solid with 12 faces.

Names above 12 are not commonly used, although they are not too complicated.
Some of the higher ones are as follows. Sometimes there is more than one possible name.

number of sides	name
13	tridecagon
14	tetradecagon
15	pentadecagon
16	hexadecagon
17	heptadecagon
18	octadecagon
19	enneadecagon
20	icosagon
30	triacontagon
40	tetracontagon
50	pentacontagon
60	hexacontagon
70	heptacontagon
80	octacontagon
90	enneacontagon
100	hectacontagon (hectogon)
1 000	chiliagon
1 000 000	miliagon

- An icosagon has 20 sides; an icosahedron is a 3-d solid with 20 faces.

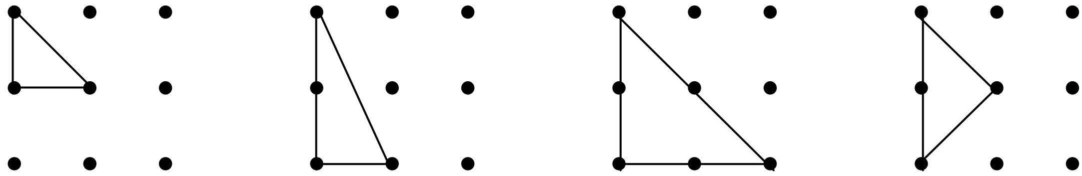
For polygons with lots of sides, you can say, for example, ***46-gon*** for a 46-sided polygon.



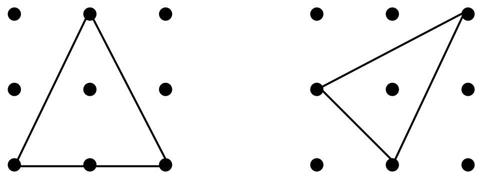
Polygons on 3 × 3 Dotty Grids **ANSWERS**

Triangles

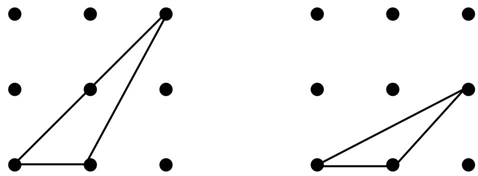
Right-angled



Acute-angled

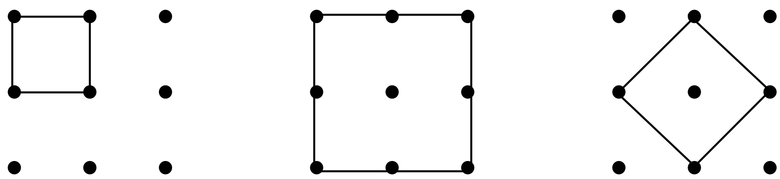


Obtuse-angled

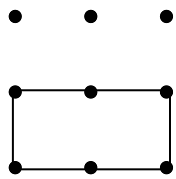


Quadrilaterals

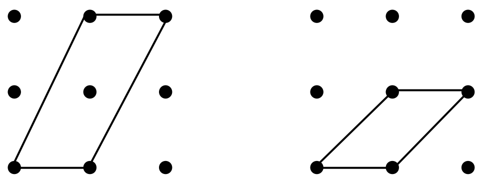
Squares



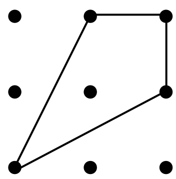
Rectangle



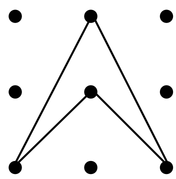
Parallelograms (rhombuses are not possible)



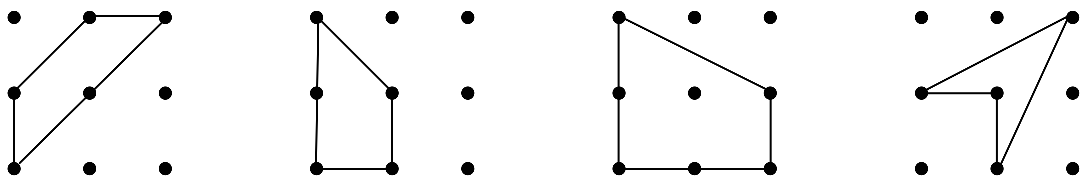
Kite



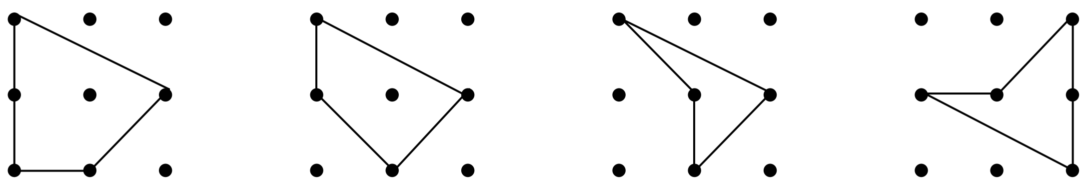
Arrowheads

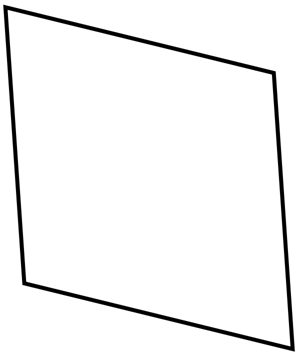
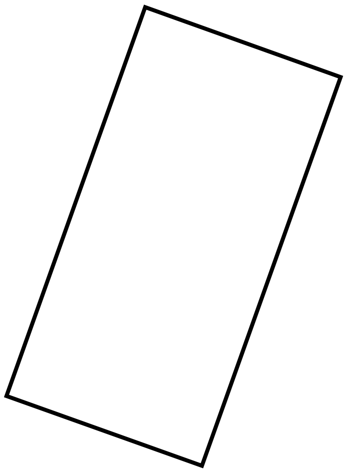
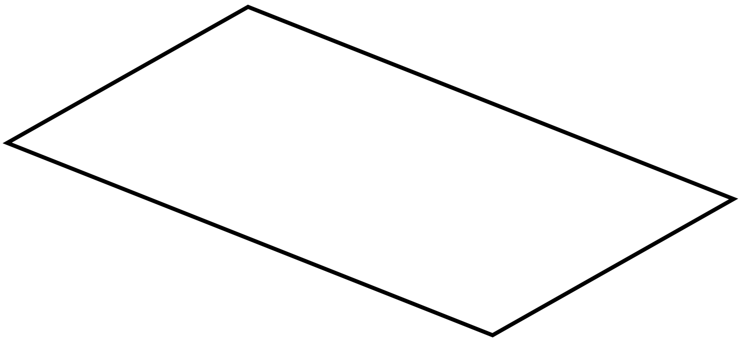
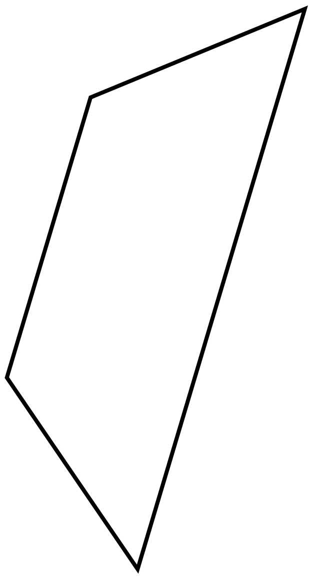
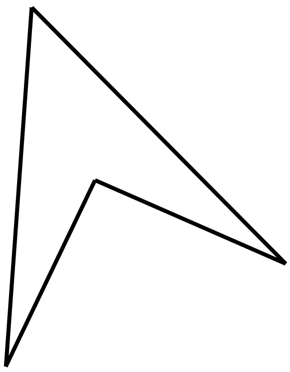
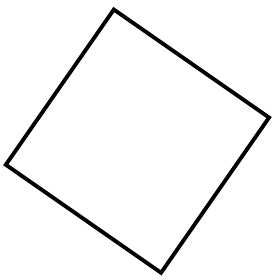
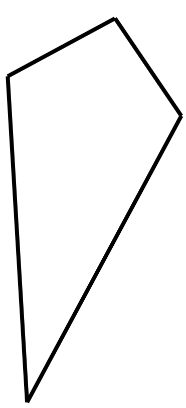


Trapeziums

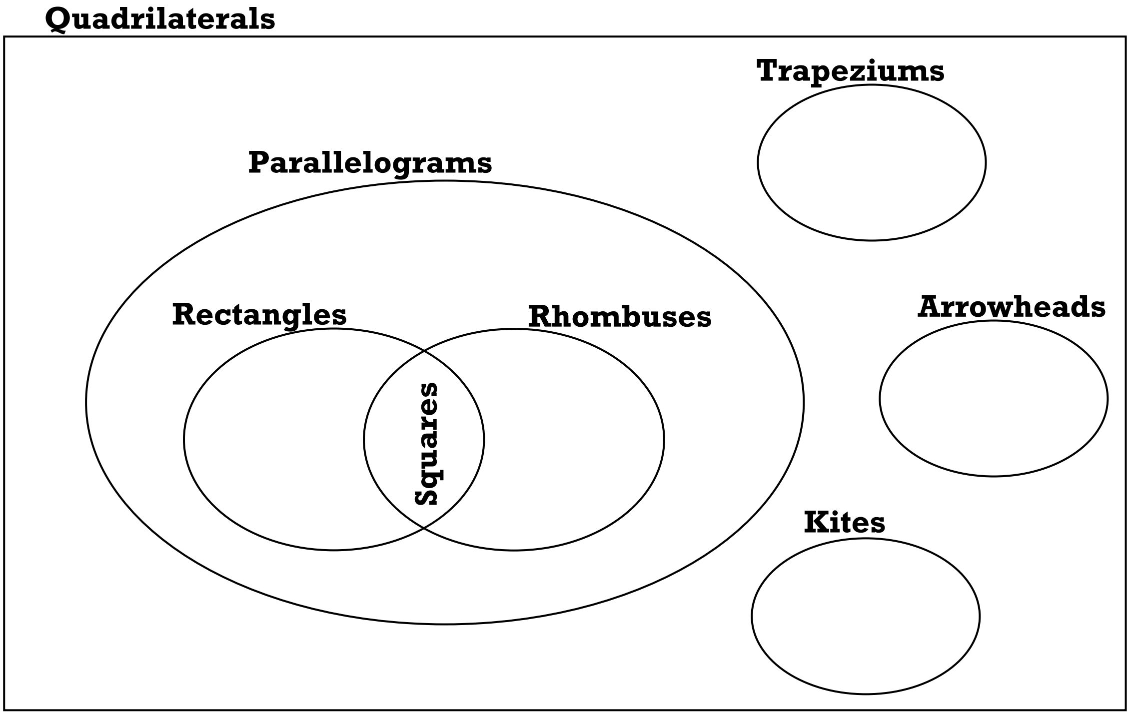


There are also 4 other quadrilaterals (below) that you can draw that don't have special names.



Classifying Quadrilaterals



There are different possible definitions: with more "inclusive" definitions, all parallelograms would count as trapeziums, and kites would include squares, rhombuses and some trapeziums.

Quadrilateral any 4-sided polygon
Parallelogram any quadrilateral with 2 pairs of parallel sides
Rectangle any quadrilateral with 4 right angles
Rhombus any quadrilateral with 4 equal sides
Square any quadrilateral with 4 equal sides and 4 right angles
Trapezium any quadrilateral with only 1 pair of parallel sides
(In an **isosceles trapezium**, the non-parallel pair of sides are of equal length.)
Kite any quadrilateral with 2 pairs of adjacent equal sides (but not all the sides equal) and no interior angle bigger than 180°
Arrowhead any quadrilateral with 2 pairs of adjacent equal sides and one interior angle bigger than 180°

Also, sometimes,
Oblong any rectangle that isn't a square

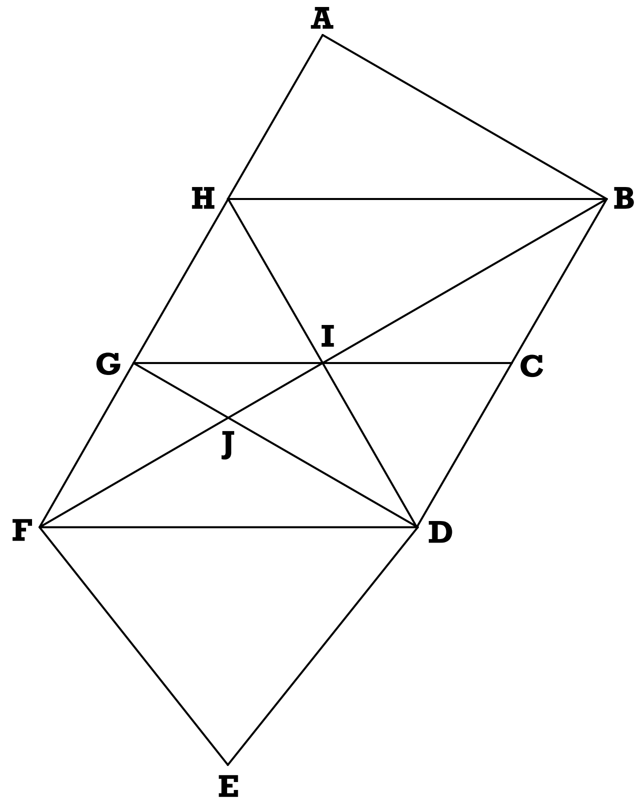
Other properties, such as lines of symmetry, orders of rotational symmetry and properties of diagonals, follow from these definitions.

Properties of Quadrilaterals (things that *must* be so for anything with that name)

	any equal sides?	**any parallel sides?**	**any equal angles?**	**anything else?**
parallelogram	2 opposite pairs	2 opposite pairs	2 opposite pairs	order 2 rot symm
rectangle	2 opposite pairs	2 opposite pairs	all 90°	2 lines of symm
rhombus	all	2 opposite pairs	2 opposite pairs	diagonals at 90°
square	all	2 opposite pairs	all 90°	4 lines of symm
trapezium	1 opposite pair if isosceles	1 opposite pair	2 adjacent pairs if isosceles	1 line of symm if isosceles
kite	2 adjacent pairs	none	1 opposite pair	diagonals at 90°
arrowhead	2 adjacent pairs	none	1 opposite pair	1 reflex angle

I have 4 equal sides and 4 right angles.	hexagon
I have 4 equal sides. The sides are **not** at right angles.	trapezium
I have 4 sides. Opposite sides are equal. Not all the sides are equal. The sides are at right angles.	parallelogram
I have 5 sides.	right-angled triangle
I have 4 sides. Only 2 sides are parallel.	scalene triangle
I have 4 sides. Opposite sides are equal. Not all the sides are equal. There are **no** right angles.	rhombus
I have 6 sides.	kite
I have 3 equal sides and 3 equal angles.	equilateral triangle
I have 3 sides. One corner is a right angle.	rectangle
I have 3 different sides.	isosceles triangle
I have 3 sides. Two sides only are equal.	pentagon
I have 4 sides. Two pairs of sides are equal. Only 1 pair of angles are equal.	square

Finding Quadrilaterals

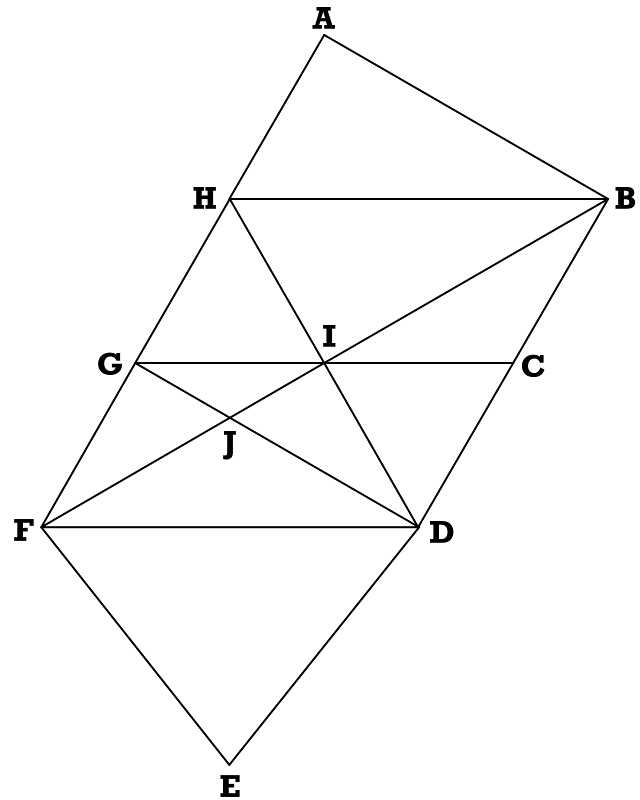


Write down the *quadrilaterals* you can find.

For example, GCDF is a parallelogram

What other polygons can you find?

Finding Quadrilaterals



Write down the *quadrilaterals* you can find.

For example, GCDF is a parallelogram

What other polygons can you find?

Imagination

Clock Polygons

What kinds of polygons do you get if you join up these times?
Ignore the minute hand and the second hand and just think about where the *hour hand* would be.

1 12.00, 4.00, 8.00
2 12.00, 3.00, 6.00, 9.00
3 12.00, 2.00, 4.00, 6.00, 8.00, 10.00
4 1.00, 6.00, 8.00
5 1.00, 3.00, 7.00, 9.00
6 12.00, 5.00, 10.00, 11.00
7 8.00, 12.00, 3.00
8 10.00, 1.00, 4.00
9 10.00, 1.00, 3.00, 6.00
10 9.00, 12.00, 3.00, 5.00, 7.00

Shape Combinations

1 I take a rectangle that isn't a square and cut a straight line along one of its diagonals. I put the two triangles that I get next to each other so that their shorter sides are touching.
What are the two possible polygons that I end up with?

2 If I do the same thing with a parallelogram, what are the two possibilities this time?

3 I put two congruent equilateral triangles next to each other so that they touch along one edge.
What shape do I get?

4 If I do the same thing with two isosceles triangles, what are the two possible shapes I could end up with?

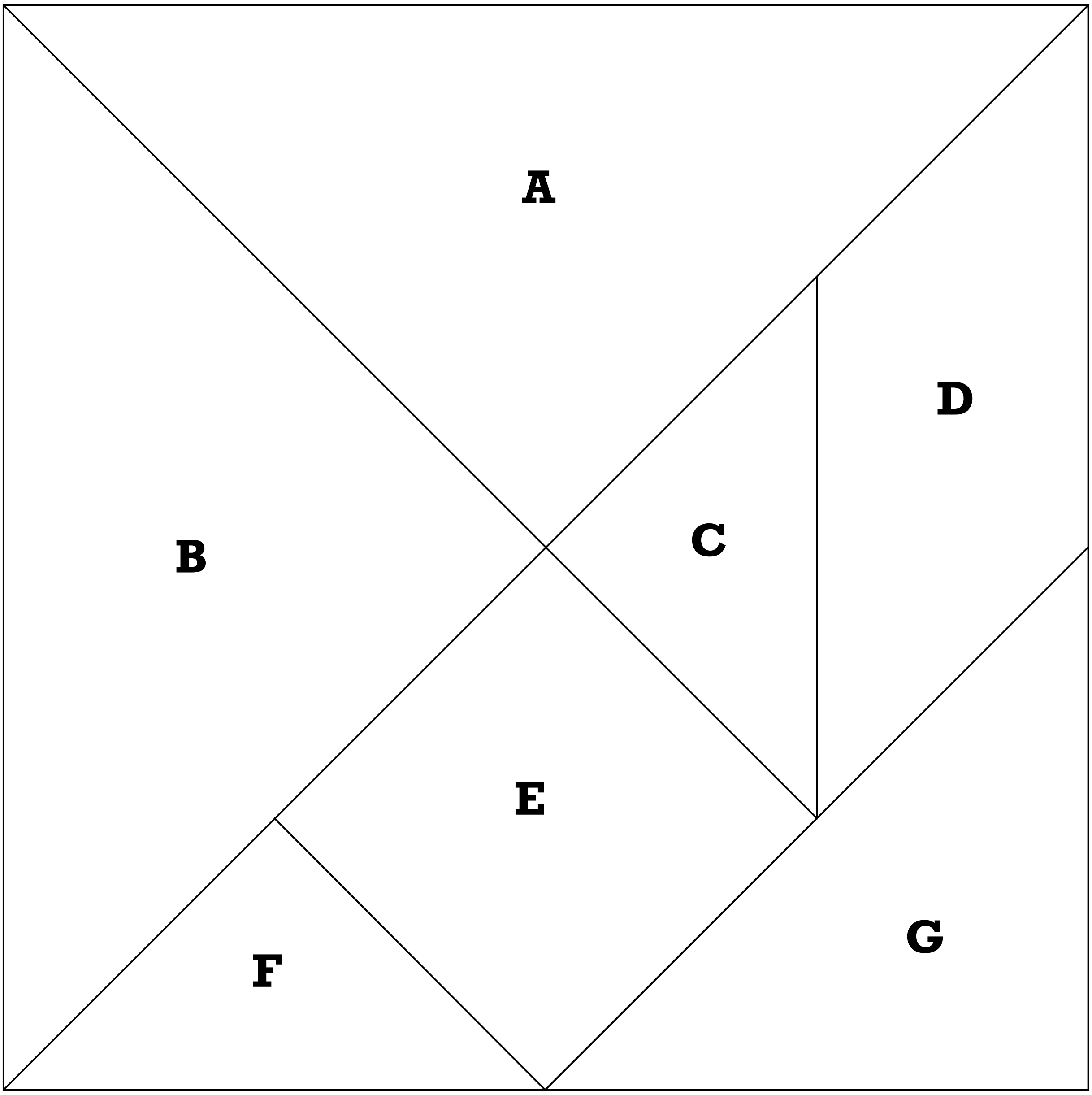
5 If I cut off one corner of a cube (this is called *truncating* a cube), what flat shape will have been created where the corner was before?

6 If I place two congruent isosceles trapeziums next to each other so that their longest sides are in contact, what shape do I get?

7 If I place two congruent isosceles trapeziums next to each other so that they touch along one of the pair of equal sides, what are the two possible shapes I could end up with?

Tangrams

Cut along all the lines so that you end up with 7 separate pieces.
You have to use all of the pieces for each puzzle.
You are not allowed to overlap any of the pieces.



2.2 Area and Perimeter

- Perimeter is easy to define: it's the distance all the way round the edge of a shape (land sometimes has a "perimeter fence"). (The perimeter of a circle is called its *circumference*.)
  Some pupils will want to mark a dot where they start measuring/counting the perimeter so that they know where to stop. Some may count dots rather than edges and get 1 unit too much.
- Area is a harder concept. "Space" means 3-d to most people, so it may be worth trying to avoid that word: you could say that area is the amount of *surface* a shape covers. (Surface area also applies to 3-d solids.) (Loosely, perimeter is how much ink you'd need to draw round the edge of the shape; area is how much ink you'd need to colour it in.)
- It's good to get pupils measuring accurately-drawn drawings or objects to get a feel for how small an area of 20 cm^2, for example, actually is.
- For comparisons between volume and surface area of solids, see section 2:10.

2.2.1 Draw two rectangles (e.g., 6 × 4 and 8 × 3) on a squared whiteboard (or squared acetate).
"Here are two shapes. What's the same about them and what's different?"
Work out how many squares they cover. (Imagine they're cm^2.) Are there any other rectangles that have an area of 24 cm^2?
Why do you think I chose 24 cm^2 and not 23?
(See related section 2.2.7.)

They're both rectangles, both contain the same number of squares, both have same area.
One is long and thin, different side lengths.

Infinitely many; e.g., 2.4 cm by 10 cm.
23 is prime, so there wouldn't be any all-integer-sided rectangles.

2.2.2 What different units can area be measured in?
When might each be appropriate?
A chart like this may help:

	÷100 → / ← ×100		÷100 → / ← ×100		÷100 → / ← ×100	
m^2		**are**		**hectare**		**km^2**

Answers: common ones such as cm^2, m^2, km^2, square miles, sq inches, sq ft, etc.

"Are" (metric) should not be confused with the word "area" or the unit "acre" (imperial):
1 acre = 4840 square yards, and
1 acre = 0.4 hectares = 40 ares.

2.2.3 **NEED** A4 1 cm × 1 cm squared paper.
Measure area of closed hand (left if right-handed, right if left-handed) and either foot (remove shoe but not sock).
Count squares which are more than half filled; ignore the others. Put a dot in the middle of squares that you've counted.

See whose foot is closest to exactly 100 cm^2!

Pupils can draw the biggest rectangle (integer sides) possible inside the shape and then use base × height to work out how many squares are there. Then just count the ones round the edge. This saves time.

2.2.4 **NEED** squared or square dotty paper.
Pick's Theorem (1859-1942).
Draw any polygon (not too big or complicated to start with) on the dotty paper. All the vertices must lie on dots.
Work out the area of the polygon.
(Break it up into simpler shapes like triangles or rectangles.)
Count the number of dots inside the shape.
Count the number of dots on the boundary (including the vertices themselves).
Look for a connection between these three quantities.

Answer:
Let i = number of dots inside the polygon and b = number of dots on the boundary (including the vertices).
Then Pick's Theorem says that $\textbf{area} = i + \frac{1}{2}b - 1$.
Proving this simple-looking formula is hard.

It turns out to have something to do with Euler's formula for polyhedra:
vertices + faces = edges + 2
(section 2.9.5).

2.2.5 Comparing Area and Perimeter.
It depends what units you use, but if you measure in "units" and "square of the same units", you can ask questions like these.
Draw shapes on 1 cm × 1 cm squared paper which have areas (a , in cm^2) and perimeters (p , cm) connected in the following ways:

The shapes must be made entirely of squares that meet along their edges.

Connection between p *and* a

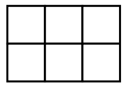
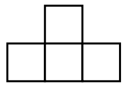
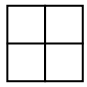
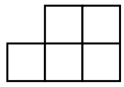
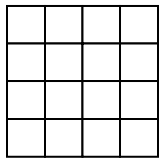
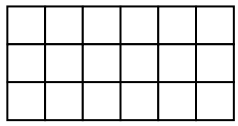
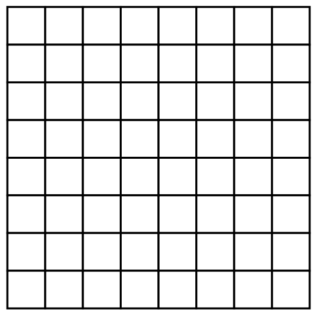
1 $p = a + 4$
2 $p = a + 6$
3 $p = 2a$, a square
4 $p = 2a$, not a square
5 $p = a$, a square
6 $p = a$, not a square
7 $a = 2p$

Why are the perimeters of these shapes always even?

If when going round the edge of a shape you move an integer number of spaces to the right, you must come back the same integer number of spaces. Likewise with up and down, so the total number of moves must be even, because it's the sum of two even numbers.

(So this applies only to rectilinear *shapes: polygons where all the interior angles are right-angles.)*

Answers: (there are many other possibilities)

		a	p
1		6	10
2		4	10
3		4	8
4		5	10
5		16	16
6		18	18
7		64	32

2.2.6 **NEED** pieces of card (see sheet) and OHP.
Area Dissections.
Demonstrate area formulas.

1. Triangle. Start with triangle 1 and label the base as b and the height as h . Introduce the two pieces of triangle 2 (which is congruent to triangle 1) and show that together with triangle 1 they make a rectangle of area bh .
2. Parallelogram. Label the base as b and the height as h . Remove the triangular end and show that it fits onto the other end to make a rectangle of area bh .
3. Trapezium. Label the height as h and the parallel sides as a and b . Introduce the other (congruent) trapezium (upside down) and show that together they make a parallelogram of area $(a+b)h$.

Pupils could use the pieces to prove the formulas to one another.

Implicit in each proof is the idea that you could always perform the same dissections and rearrangements whatever the precise shape.

So triangle area = $\frac{1}{2}bh$ *.*

So parallelogram area = bh *.*

So trapezium area = $\frac{1}{2}(a+b)h$ *.*

(Notice that if $a = 0$ *(or* $b = 0$ *), the shape becomes a triangle and the area formula becomes* $\frac{1}{2}bh$ *, as it should.)*

2.2.7 Draw a rectangle with an area of 24 cm^2.
(This builds on section 2.2.1.)
Work out its perimeter and write it inside.
Repeat.
What are the biggest and smallest perimeters you can find?
You are not allowed to change the total area.

Answers:
Rectangles with integer *sides and area 24 cm^2:*

rectangle	perimeter	rectangle	perimeter
1 × 24	50 (max)	2 × 12	28
3 × 8	22	4 × 6	20 (min)

If non-integer values are allowed, then the smallest perimeter would come from the square, which has sides $\sqrt{24} = 4.9$ units and a perimeter of 19.6 units.
The largest would come from a very long thin rectangle δ by $\frac{24}{\delta}$, where δ is small. The perimeter would be $2(\delta + \frac{24}{\delta})$, which tends to infinity as δ gets smaller and smaller. So you could make the perimeter as large as you like.

With integer *sides, the smallest perimeter belongs to the rectangle most like a square. The largest perimeter comes from the rectangle of width 1 cm.*

Why do you think I chose 24 cm^2 and not, say, 23 cm^2 for the area?

24 has lots of factors; 23 is prime.

What if I fix the *perimeter* at 24 cm, and ask for the biggest and smallest *areas* you can make? Still only rectangles are allowed.

Rectangles with integer sides and a perimeter of 24 cm:

rectangle	area	rectangle	area
1 × 11	11	2 × 10	20
3 × 9	27	4 × 8	32
5 × 7	35	6 × 6	36

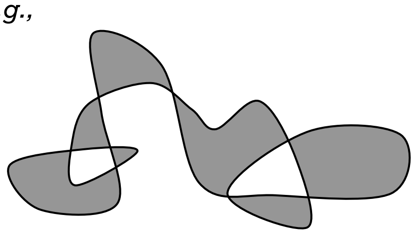
Again, the largest area for a given perimeter (or smallest perimeter for a given area) comes from the square.
Areas as small as you like come from rectangles δ by $12 - \delta$, where δ is small, and have area $\delta(12 - \delta)$, which tends to zero as δ gets smaller and smaller.

You may need to hint that a square is a rectangle and so is allowed without giving the game away!

(This investigation is extended in section 2.6.9)

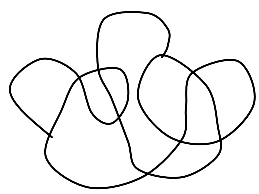
2.2.8 How many colours do you need to colour the countries on a map?
Draw a pattern (not too complicated) without taking your pen off the paper. You can cross over yourself, but you must finish at the point you started.
If you want to colour it in so that always when two areas have a side in common they are different colours, how many colours do you need? (It's OK for the same colours to touch at a point, just not at a side.)

Answer: only 2 colours needed
e.g.,



What if you don't finish at the point you started?

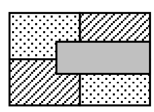
e.g.,



This time you need up to 3 colours.

Can you find a design that needs more than 3 different colours?
(You are allowed to take your pen off the paper during the drawing, now.)

Some designs need 4 colours; e.g.,



(Remember that the outside needs colouring – white in this case.)

Any "map" can be coloured with at most 4 colours. This "Four Colour Theorem" is very hard to prove, but was eventually done using computers. (This was the first major theorem to be proved using a computer.)

2.2.9 Polyominoes (1 cm × 1 cm or 0.5 cm × 0.5 cm squared paper is useful).
Start with dominoes – there's only one possible flat shape you can make by placing two squares next to each other. They mustn't overlap and they may only touch edge-to-edge.
What about triominoes (3 squares), and so on? How many different ones can you find?

The number of squares a is the area, and for $a \leq 3$ the perimeter p is the same for each polyomino of that size and equal to $2(a+1)$.

The 12 pentominoes are sometimes referred to by the capital letters they look most like: FILNPTUVWXYZ.

You can do a similar thing with equilateral triangles (the shapes are called *polyiamonds*) and with hexagons (*polyhexes*).
How many of those can you find?

The three tetriamonds are the three possible nets for a tetrahedron.

Polyhexes (made up of regular hexagons) have relevance in organic chemistry because they give the number of possible isomers of some of the aromatic hydrocarbons. For example the three trihexes correspond to anthracene, phenanthrene and phenalene (all $C_{14}H_{10}$), where the hexagons are rings of carbon atoms with hydrogen atoms attached.

Answer:
The numbers grow very quickly.

no. of squares	name	no. of polyominoes
1	square	1
2	domino	1
3	triomino	2
4	tetromino	5
5	pentomino	12
6	hexomino	35
7	heptomino	108
8	octomino	369
9	nonomino	1285

There is no simple pattern to these numbers. (From 7 onwards some of the polyominoes contain holes.)

no. of triang's	name	no. of polyiamonds
1	equil. triangle	1
2	diamond	1
3	triamond	1
4	tetriamond	3
5	pentiamond	4
6	hexiamond	12
7	heptiamond	24
8	octiamond	66
9	enneiamond	160

no. of hex'ns	name	no. of polyhexes
1	hexagon	1
2	dihex	1
3	trihex	3
4	tetrahex	7
5	pentahex	22
6	hexahex	82
7	heptahex	333
8	octahex	1448
9	enneiahex	6572

2.2.10 A politician claims that the world isn't overcrowded at all. He says that every person in the world could have an average-sized house (and garden) and the whole lot would fit into California.
What do you think?
What data would you need to test his claim?

The area of California is about 4×10^5 km², so he is about 7 times out.

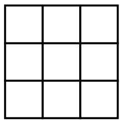
My aunt says you could fit everyone in the world onto the Isle of Wight if they lined up shoulder-to-shoulder (all standing on the ground). Is that possible?

Again, my aunt is exaggerating but not all that much.

Answer: He may be exaggerating, but not that much. Assuming that there are about 6.5×10^9 people in the world, and each average-sized property measures about 20 m × 20 m, then the total area needed = $6.5\times10^9\times20\times20$ = 2.6×10^{12} m².
Since 1 km² = 10^6 m², this is only about 3×10^6 km². The area of the USA is about 9.5×10^6 km², so this is about $\frac{1}{4}$ of that.

To make the sums easier, we'll give everyone 0.5 m by 0.5 m = 0.25 m², which should be enough room.
Total area = $6.5\times10^9\times0.25$ m² = 1625 km². This is about 40 km by 40 km, or 635 square miles. This is about four times the area of the Isle of Wight (about 150 square miles).

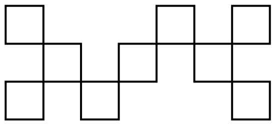
2.2.11 Display cabinets.
A museum curator wants to arrange her glass display cabinets so that visitors can view the exhibits. She has 9 square cabinets. What is the best arrangement?



Why would a 3 × 3 square arrangement be a bad idea?

Draw the maximum perimeter arrangement if the cabinets must be connected side to side (not corner to corner).
Find a connection between the number of cabinets and the maximum perimeter.

If the cabinets could touch only at the corners, the maximum perimeter would be $4n$*; e.g.,*

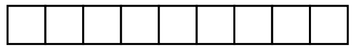


In a different room she wants to use cabinets shaped like equilateral triangles (when viewed from above). Find a formula for the maximum perimeter when she uses different numbers of these cabinets.
What about other regular polygon cabinets?

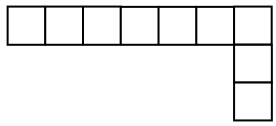
Answers:
The cabinets have to touch along the sides.

With a 3 × 3 arrangement, it would be very difficult to see the cabinet that is in the middle, and the others could only be viewed from 1 or at most 2 sides.

You want to get the maximum perimeter so that there's the maximum number of sides people can view from.



Always put the cabinets in a single line. In fact simple turns don't affect the perimeter; e.g., this arrangement has the same perimeter (20) as the L-shape below.



If n *is the number of cabinets, then the maximum perimeter is* $2n+2$*.*

With n *equilateral triangles, the maximum perimeter is* $n+2$*; for regular pentagons, the maximum perimeter is* $3n+2$*.*
For an r*-sided regular polygon, the maximum perimeter is* $(r-2)n+2$*.*

2.2.12 Areas of Parallelograms (see sheet).
Investigating what controls the area of a parallelogram.

Answers (in square units):
A. 10; B. 15; C. 3; D. 6; E. 9; F. 12;
G. 10; H. 10; I. 10; J. 10.

2.2.13 **NEED** A4 1 cm × 1 cm squared paper.
"Design a Zoo" (see sheet).

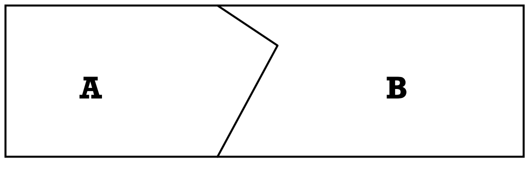
Makes good display work and combines different aspects of maths. An enjoyable task.

2.2.14 A ream of A4 paper is described as 80 g/m^2.
What is the mass of an individual sheet?
How much does the whole packet weigh?
How many could I send first class?
(Assume that the envelope weighs 10 g.)

A "ream" of paper is 500 sheets.

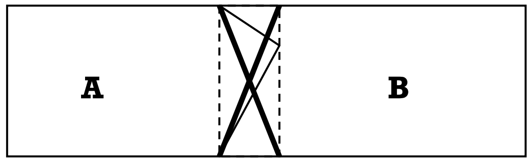
Answer:
1 sheet weighs 0.21 × 0.297 × 80 = 5 g (approx)
500 sheets weigh about 2.5 kg.
If the maximum mass for the cheapest postage rate is, say, 60 g, then we could put 10 sheets of paper in the envelope. (This ignores the mass of the stamp and any ink.)

2.2.15 Two people, Alison and Billy, own some land as shown below.



They want to replace the V-shaped fence with a straight line so that their plots will have a more convenient shape, but they must keep the same amounts of land each.
Where should the line go?
(The land is equally good everywhere.)

Answer:



Draw in the parallel dashed lines as above. Then the new fence should be either of the thick black lines, because the area of the obtuse-angled triangle is the same as the areas of the right-angled triangles with the same base and the same height, so A can have one of those instead with no change in area.

2.2.16 196 soldiers are marching in a square arrangement. The soldiers at the edge of the square have to carry a flag. How many of these "outside" soldiers are there?
How many "inside" (non-flag-carrying) soldiers are there?

Altogether, $4n-4+(n-2)^2$

$=4n-4+n^2-4n+4$

$=n^2$

as it should do.

Answers:
For n^2 *soldiers, there will be* $4n-4$ *on the perimeter (4 sides of* n *soldiers makes* $4n$*, but the 4 at the corners get counted twice because they each belong to two sides, so we have to subtract those 4).*
The inside soldiers make a square arrangement of $(n-2)^2$.

So for 14^2 *(196) soldiers, the number outside is 52 and the number inside is 144, so 52 flags are needed.*

2.2.17 **NEED** tape measures, possibly other things as well. Estimate the surface area of a human being.

Practical methods: e.g., wrap someone up in newspaper; use sticky tape and remove the wrapping by cutting carefully with scissors so that when flattened out it approximates the area.

Theoretical methods: e.g., ignore hands, feet, etc., and treat the human body as a sphere on top of a cuboid with two identical cylindrical arms and two bigger identical cylindrical legs. (Different pupils may decide on different assumptions.)

(See similar task in section 2.10.14.)

Answer: the value is not too important – it's the process adopted that matters – but suggested values are given below.

Values will obviously depend on the size of the pupils.

Theoretical approximation:
Head: $4\pi r^2 = 4\pi 10^2$ = *1300 cm*^{2}*;*
Trunk:
2 × (50 × 50 + 20 × 50 + 50 × 20) = 9000 cm^2*;*
Arms:
2 × $2\pi rl$ *= 2 × 2 × 3.14 × 4 × 50 = 2500 cm*^{2}*;*
Legs:
2 × $2\pi rl$ *= 2 × 2 × 3.14 × 6 × 80 = 6000 cm*^{2}*;*
So total estimate = 18 800 cm^2 *= 2 m*^{2} *approximately, which seems sensible.*
(Lungs have surface area of about 100 m^2*, and the intestines about 300 m*^{2}*!)*

2.2.18 Estimate how many tins of paint you would need to paint this classroom.

Before you start decide if there's anything you need to ask me?

Again, the thinking that pupils go through is much more important than the final estimate.

e.g., how many coats of paint?; are we painting behind the cupboards?; are we doing the ceiling? are we doing the door? etc.

You could state that an "average" tin of paint will cover about 15 m^2*.*

2.2.19 Heron's Formula (Heron of Alexandria, about AD 10-75) (see sheet).
A formula for calculating the area of a triangle given only the lengths of the sides.

Also known as Hero's Formula.

This formula probably should be more widely known and used.

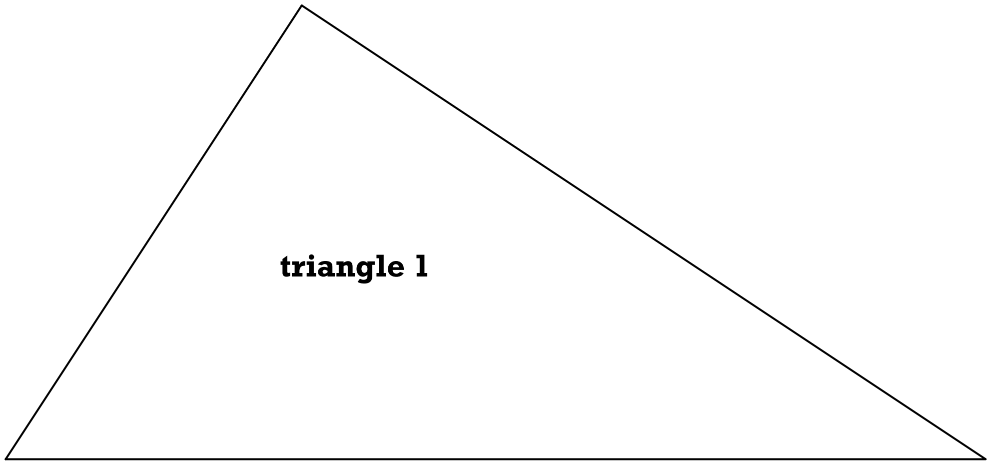
2.2.20 Tolstoy (1828-1910) wrote a short story called "How Much Land Does A Man Need?" in which a peasant man called Pahóm is offered some land at a price of "1000 roubles *per day*".
It turns out that he can have as much land as he can go round by foot between sunrise and sunset, but he must finish back where he started before the sun goes down.
What would your strategy be if you wanted to get as much land as possible?

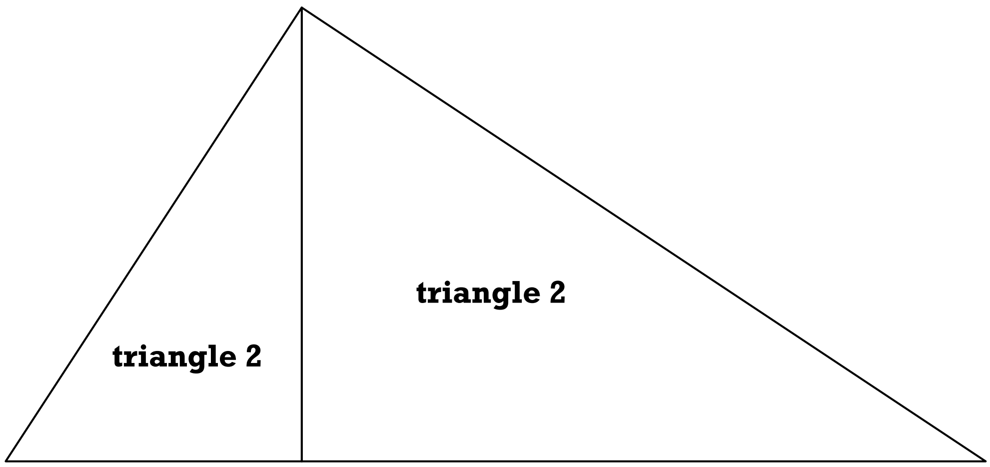
(It's an interesting story pupils may like to read – not too long and with a twist at the end – and impressive to say you've read some Tolstoy!)

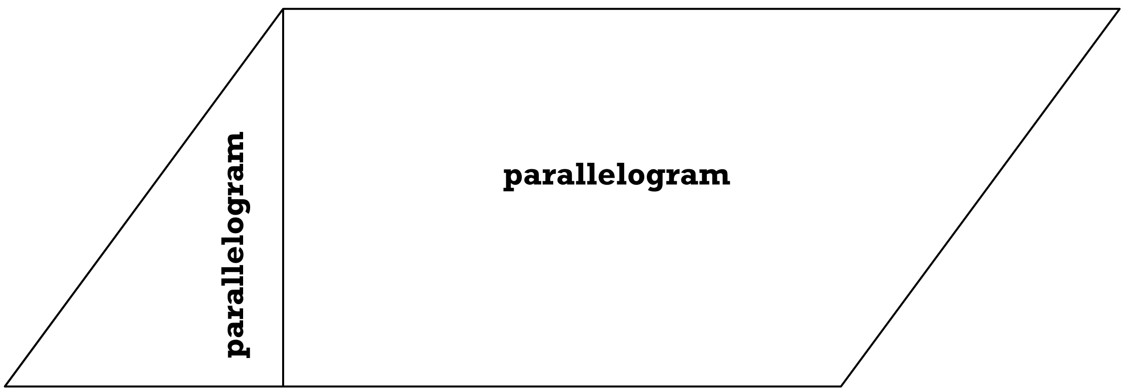
What shape path would you take? Would you run or walk? Would it be more efficient in the long run to take breaks? What if you saw some particularly good land? What would you do about hills?

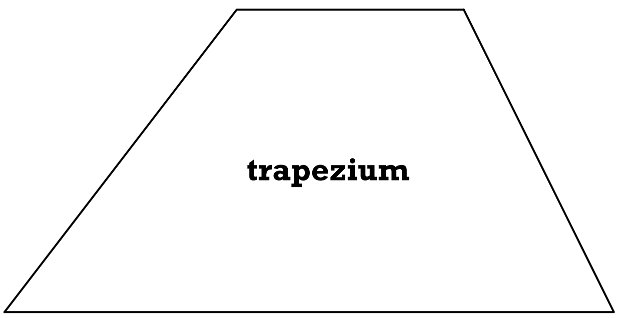
Area Dissections

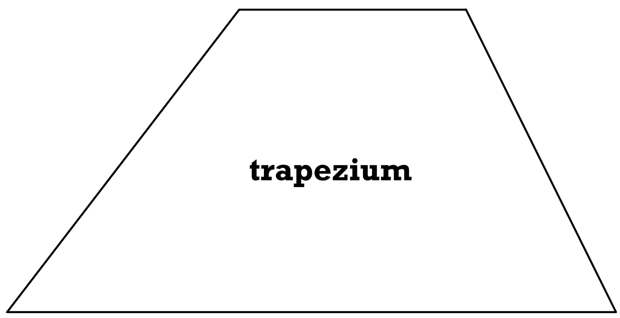
Photocopy onto card and cut along *all* the lines. Keep the pieces in an envelope.
Use to demonstrate area formulas (see notes).











Areas of Parallelograms

Draw some axes from 0 to 8 horizontally and vertically.
One set should do (with a bit of overlapping) for A to F, and another one for G to J.

Plot each of these parallelograms.
Work out their areas by breaking them into triangles or rectangles.
Record your results.

A	(1, 6)	(2, 8)	(7, 8)	(6, 6)
B	(0, 6)	(5, 6)	(7, 3)	(2, 3)
C	(7, 5)	(8, 6)	(8, 3)	(7, 2)
D	(0, 2)	(0, 4)	(3, 2)	(3, 0)
E	(4, 0)	(3, 3)	(6, 3)	(7, 0)
F	(1, 5)	(3, 7)	(7, 5)	(5, 3)
G	(0, 6)	(1, 8)	(6, 8)	(5, 6)
H	(0, 4)	(2, 6)	(7, 6)	(5, 4)
I	(0, 2)	(3, 4)	(8, 4)	(5, 2)
J	(1, 0)	(0, 2)	(5, 2)	(6, 0)

What things affect the area of a parallelogram and what things make no difference?

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E	(4, 0)	(3, 3)	(6, 3)	(7, 0)
F	(1, 5)	(3, 7)	(7, 5)	(5, 3)
G	(0, 6)	(1, 8)	(6, 8)	(5, 6)
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I	(0, 2)	(3, 4)	(8, 4)	(5, 2)
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What things affect the area of a parallelogram and what things make no difference?

Design a Zoo!

(Teachers' Notes)

- There will be **10 animals** to house in the zoo.

- Each animal will have a separate **cage**.
  All the cages will be made of 6 squares but arranged differently so that each cage is a different shape.

- Take an A4 piece of 1 cm × 1 cm squared paper.
  This will be your plan for the zoo.
  Use a scale of 1 cm to 1 m and draw the cages, spreading them out over the page.
  Each cage will have an area of 6 m^2.
  Label which cage is for which animal.

- The floor material for the cage will cost £500 per m^2. (All prices include labour!)
  So for 10 cages you will have to spend 10 × 6 × 500 = £30 000.
  Head another sheet of paper "Accounts" and record this cost.
  Show how you worked it out.

- Every cage needs **fencing** round the edge.
  This costs £200 per m.
  Calculate the perimeter of each cage – they won't all be the same.
  Find the total of all the perimeters and multiply this by £200 to find the total fencing cost.
  Put all of this on your accounts sheet.

- Your **budget** for designing the whole zoo is **£250 000**.
  You cannot go over this.

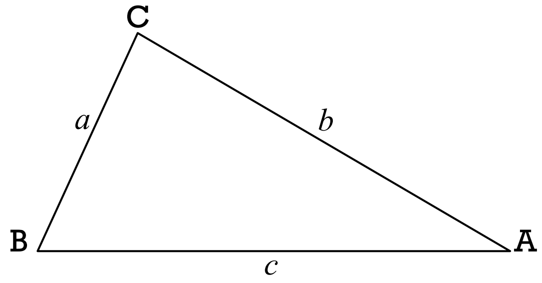
- **Things to add:**
  - **path** to take visitors round the zoo so they can look into each cage.
    Design it and work out how much it will cost.
    Cost = £100 per m^2.
  - **signs** to show the visitors what's where
    Cost = £50 each
  - **car park**
    Cost = £50 per m^2 of gravel
  - **trees**
    Cost = £25 each
  - **toilets**
    Cost = £7 500
  - **café**
    Cost = £25 000

- What other things could you add? Your teacher will give you a quotation!
  You must keep within budget!
  How much would you charge people to visit the zoo?

- Do you think that the cages would be large enough?
  Do you think that the prices are realistic?

Heron's Formula

Consider any triangle in which the lengths of the sides a, b and c are known and we wish to find the area.



We can use the cosine rule to work out one of the angles (C) and then use the formula **area** $= \frac{1}{2}ab\sin C$ to find the area.

Using the cosine rule, $c^2 = a^2 + b^2 - 2ab\cos C$, so $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$.

Using the identity $\sin^2 C + \cos^2 C \equiv 1$, we can find an expression for $\sin C$, and we get

$$\sin C = \sqrt{1 - \cos^2 C} = \sqrt{1 - \left(\frac{a^2 + b^2 - c^2}{2ab}\right)^2} = \sqrt{\frac{4a^2b^2 - (a^2 + b^2 - c^2)^2}{4a^2b^2}}.$$

Now using **area** $= \frac{1}{2}ab\sin C$ we get

$$\textbf{area} = \tfrac{1}{2}ab\sqrt{\frac{4a^2b^2 - (a^2 + b^2 - c^2)^2}{4a^2b^2}} = \tfrac{1}{4}\sqrt{4a^2b^2 - (a^2 + b^2 - c^2)^2}.$$

Factorising the difference of two squares inside the square root sign gives

area $= \frac{1}{4}\sqrt{\{2ab + (a^2 + b^2 - c^2)\}\{2ab - (a^2 + b^2 - c^2)\}}$, and rearranging and factorising again gives **area** $= \frac{1}{4}\sqrt{\{(a+b)^2 - c^2\}\{c^2 - (a-b)^2\}}$.

Again using the difference of two squares we get

$$\begin{aligned}
\textbf{area} &= \tfrac{1}{4}\sqrt{(a+b+c)(a+b-c)(a-b+c)(c-a+b)} \\
&= \sqrt{\left(\frac{a+b+c}{2}\right)\left(\frac{a+b-c}{2}\right)\left(\frac{a+c-b}{2}\right)\left(\frac{b+c-a}{2}\right)} \\
&= \sqrt{\left(\frac{a+b+c}{2}\right)\left(\frac{b+c-a}{2}\right)\left(\frac{a+c-b}{2}\right)\left(\frac{a+b-c}{2}\right)} \\
&= \sqrt{s(s-a)(s-b)(s-c)}
\end{aligned}$$

where $s =$ the semi-perimeter $= \dfrac{a+b+c}{2}$.

This formula **area** $= \sqrt{s(s-a)(s-b)(s-c)}$ for the area in terms of the semi-perimeter s and the sides a, b and c is called **Heron's Formula**.

2.3 Circles

- Note that "circumference", "diameter", "radius", etc. can refer either to the lines themselves ("things") or to the lengths of those lines; e.g., the length of a radius is often just called the radius.
- All circles are mathematically similar (like all squares, for example, but unlike, say, all right-angled triangles).
- You can write the area formula as $r^2\pi$ to avoid the danger of calculating $(\pi r)^2$ instead of πr^2.
- There are lots of definitions to grasp: an **arc** is part of the circumference of a circle; a **chord** is a straight line joining two points on the circumference (a diameter is a chord that goes through the centre); a **tangent** is a straight line touching the circumference at one point only; a **sector** is the area between an arc and two radii (a **semicircle** is a sector which is half a circle; a **quadrant** is a sector which is a quarter of a circle); a **segment** is the area between a chord and an arc. (Segments and sectors are easy to muddle up – a semicircle is both.) Circumference is just the perimeter of a circle.
- Material using Pythagoras' theorem in the context of circles is in section 2.7.

2.3.1 **NEED** string or tape measures and "round" objects or "Circles" sheet, callipers if you have them.
Practical Investigation: we're going to discover something interesting about circles. Bring in or find circular objects (or objects with circular cross-section): dinner plate, clock, football, window, tiles, rubber, pencil sharpener, food tin, cup, marker pen, bin, sticky tape, someone's arm.
Measure the circumference and the diameter. Is there a connection between these two amounts?

Divide the circumference by the diameter (use the same units). What do you get?

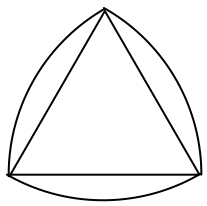
Pupils will realise that you can't get π very accurately by this method!

This leads to a value of π of about 3 (or "3 and a bit"). It's nice to demonstrate this "3 and a bit" if there's a fairly large (> 1 m diameter) circular object in school. Wrap the string around the outside and cut it the length of the circumference. Measure with it across the middle 3 times, and "a bit" is left over. (This can be quite a memorable demonstration.)

It's pretty amazing that $c = \pi d$ works regardless of scale; e.g., for a microscopic water drop or a giant star or planet's orbit.

In practice, it's usually easier to measure the diameter than the circumference, because straight lines are easier to measure accurately, but sometimes you can't "get at" the diameter (e.g., a pipe), and then it's useful to be able to calculate *the diameter from the circumference.*

2.3.2 What is it about circles that makes them good for wheels? Is there any other kind of shape that would do?



This shape is also used for drills that drill "square" holes (almost square – the corners aren't quite right).

Which other polygons can you make curvy versions of like this?

These are sometimes called "rolling polygons".

Answer: It's their constant "width" (diameter) regardless of orientation, so that whatever is travelling on top is always the same height off the ground.

Other shapes do that; e.g., a Reuleaux triangle (Franz Reuleaux, 1829-1905), formed by adding arcs to each side of an equilateral triangle (radius the same as the lengths of the sides of the triangle) – see left.
Although something resting on top would be carried horizontally, the centre *of the wheel wobbles up and down, so it wouldn't be any good on an axle.*

It works for all the regular polygons that have an odd *number of sides; seven-sided versions are used for 20 p and 50 p coins. Their constant width regardless of orientation helps in slot machines.*

2.3.3 Could you describe a circle over the telephone to someone who didn't know what one was? (Imagine an alien who doesn't know, for example, what a football looks like or what we mean by "round".)

This is quite hard, although it seems like such a simple thing! You could say "a set of all the possible points that are a certain fixed distance from a fixed point in 2 dimensions."
(This would define a sphere in 3 dimensions.)

2.3.4 **NEED** sheets of circles drawn on 1 cm × 1 cm squared paper (containing circles of radius 6 cm/7 cm and 5 cm/8 cm).
We're looking for a connection between the radius of a circle and its area. Count 1 cm^2 squares (count the square if the circle covers half or more of the square, otherwise ignore it). Make a table of the radius versus area and look for a pattern.

Especially with the larger circles, it is sensible to mark off a big square of 1 cm^2 squares in the middle of the circle and find its area by multiplication, as that saves counting every single 1 cm^2. Then you can count the ones round the edge and add the two amounts.

You could divide up the work among the class so that, perhaps in groups of 2 or 3, pupils work on a couple of different-sized circles. The teacher can collect all the results on the board (doing some kind of average of the results people contribute, rejecting anything way out). Someone could try the 3 cm, 4 cm and 9 cm radius circles as well.

Calculated results (typically you get within a couple of cm^2 experimentally):

radius (cm)	*area (cm^2)*
3	*28.2*
4	*50.2*
5	*78.5*
6	*113.1*
7	*153.9*
8	*201.1*
9	*254.5*

A clue to help with seeing the connection is to square the radius numbers and then look for a pattern. Make predictions and check.

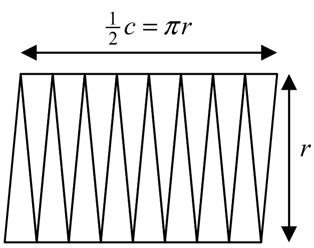
Pupils can plot the results on a graph (area on the vertical axis, radius on the horizontal).

Should get a parabola curve.

Using compasses and 1 cm × 1 cm squared paper, try to draw a circle with an area of exactly (or as near as you can) 100 cm^2. Use the graph to decide what the radius ought to be. Check by counting the squares.

Answer: the exact radius needed is $\sqrt{\frac{100}{\pi}} = 5.64$ *cm.*

2.3.5 **NEED** scissors, glue and 5 cm radius circles (draw with compasses). Cut out the circle and divide it roughly into sixteenths (8 lines), like cutting up a cake. Cut along all the lines so that you get 16 sectors of the circle.
Arrange them into an approximate "rectangle/parallelogram".



So the area is $\pi r \times r = \pi r^2$.

A circle this size produces a "rectangle" that fits nicely on an approximately A5 exercise book page.

This is more than an approximation, because we can imagine splitting up the circle into 32, 64, 128, etc. pieces; in fact, as many as we like, so we can make a shape which is as close to a rectangle/parallelogram as we like.

The more pieces we use, the more valid this argument becomes, so the area of the parallelogram gets closer and closer to the true area of the circle, so πr^2 *must be the true area of the circle.*

2.3.6 Function machines are useful for managing conversions between A, r, d and c.
It's easy to make up questions and put the values into a table.

See sheet.

2.3.7 The Number π is a *transcendental* number (it doesn't satisfy any polynomial equation with integer co-efficients). You can't write it as a fraction using integers (it's *irrational*). The decimal digits go on for ever and never go back to the beginning and repeat. Everyone's telephone number and credit card number is in there somewhere! *(It has not actually been proved that the digits of π are "random" in the sense that every possible combination of digits of a given length comes up equally often, but it is very probably true.)*

Ways of Calculating π:

1. $\frac{\pi^2}{6}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+...$, but this converges very slowly (145 terms to get 2 dp);
2. $\frac{\pi}{4}=\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-...$, which also converges slowly;
3. $\frac{\pi^4}{90}=\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+...$, which converges quickly (only 3 terms to get 2 dp).
4. $\pi=2\times\frac{2}{1}\times\frac{2}{3}\times\frac{4}{3}\times\frac{4}{5}\times\frac{6}{5}\times\frac{6}{7}\times...$, called Wallis' product (John Wallis, 1616-1703).

Some pupils may find these interesting:

$\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,dx=\sqrt{2\pi}$ *and* $e^{i\pi}+1=0$

All transcendental numbers are irrational. The opposite of transcendental is "algebraic".

Other transcendental numbers include e, e^{π}, $\ln 2$ and $\sqrt{2}^{\sqrt{2}}$. No-one knows if e^e, π^{π} or π^e are transcendental.

(See sheet for the first 10 000 or so digits of π – you can photocopy back-to-back onto card and pass around the room: pupils may try to find their phone numbers!)

2. is the Maclaurin (1698-1746) series for $\tan^{-1}1$, but it is also called the Leibniz (1646-1716) or Gregory (1638-1675) series.

You could try these series on a spreadsheet. There are many other ways of calculating π. To get a large number of digits, you need more efficient processes than these.

π is a letter in the Greek alphabet (does anyone know Greek?). It has nothing to do with pies often being circular or pie charts or Pythagoras's Theorem!

2.3.8 Imagine a cable lying flat on the ground all the way round the equator and back to where it started. If instead you wanted to support the cable all the way round on poles 10 m high, how much more cable would you need? (We have to ignore the existence of the sea!)

(You don't need to know the radius of the earth, but it's 6.4×10^6 m, and you can provide it as unnecessary information if you like!)

An alternative version of this is the following puzzle: A businessman sets out on a journey, eventually returning to the place where he started. He claims that during his trip his head has travelled 12.6 m further than his feet have. How can that be possible?

Answer:
Additional cable = $2\pi(r+10)-2\pi r$ which is just $2\pi 10 = 62.8$ m.
Much less than people generally expect.

The radius of the earth doesn't matter (it would be the same extra amount putting cable 10 m around a 2p piece), because for a larger circle you need a smaller proportion of a bigger amount; for a smaller circle a bigger proportion of a smaller amount.

He has been once round the equator and his height is 2 m.

2.3.9 Imagine a circular coin of radius r rolling round the edge of a square with perimeter p so that it never slips. How far does the centre of the coin move when the coin goes round the square once?

What if you rolled the coin round a different polygon (still with total perimeter p)?

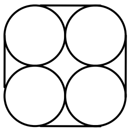
What if you rolled it round an identical coin?

Answer: $p+2\pi r$ units. (It follows the edges of the polygon but also, at each vertex, the centre moves in an arc. By the time it gets back to the beginning it's turned through 360°, and that's where the extra $2\pi r$ comes from.)

Same result. The polygon doesn't have to be regular, although it does need to be convex.

Effectively, the same result, with $p=2\pi r$, so the total is $2\pi r+2\pi r=4\pi r$ units.
(The centre just moves round a circle with total radius $2r$, so you can calculate $2\pi(2r)=4\pi r$, the same answer.)

2.3.10 Four large pipes, each of 1 m diameter, are held tightly together by a metal band as shown below. How long is the metal band?



What if instead there are only three pipes?

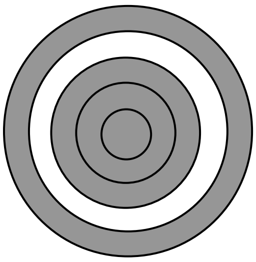
What if there are n pipes?

Answer:
There are four quarter-circle arcs (one on each pipe) with a total length of $2\pi r = \pi$ *and four straight pieces with a total length of* $4 \times 2r = 4$*, so the total length of the metal band is* $4 + \pi$ *metres = 7.14 m.*

By the same argument, length = $3 + \pi$ *m.*

For n *pipes,* $n + \pi$ *metres (*$n > 1$*).*
If $n = 1$*, it is just* π *metres.*

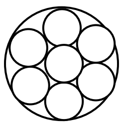
2.3.11 (You can draw this reasonably well on a squared whiteboard.)
Which shaded area is bigger (could use different colours), the outer or the inner?



Answer:
Pupils may guess that they're the same, although many people think that the middle three rings look bigger.
Area of the outer ring = $\pi(5^2 - 4^2)$
= $\pi 3^2$ *= area of first three rings.*

If it were a "dartboard" it probably would be easier to hit the middle three rings than the outer one, because although the areas are the same the outer one has a very thin width. (Imagine trying to hit a 4 cm × 4 cm square; that would be much easier than a 1 cm × 16 cm rectangle, although they have equal areas.)

2.3.12 Imagine a circular sheet of metal of diameter 6 m. What percentage of the metal will be wasted if you cut out two circles, each of diameter 3 m?
How many 2 m diameter circles can you cut out of the original 6 m diameter sheet? What would be the percentage wasted this time?



Answer:
$\frac{\textbf{area used}}{\textbf{total area}} = \frac{2 \times \pi (1.5)^2}{\pi 3^2} = \frac{1}{2}$*, so 50% is wasted.*

Answer: 7 circles is the maximum (see drawing on the left)

With 2 m circles, $\frac{\textbf{area used}}{\textbf{total area}} = \frac{7 \times \pi 1^2}{\pi 3^2} = \frac{7}{9}$*, so only* $\frac{2}{9}$ *or 22% is wasted now.*

2.3.13 A washing machine has a drum of diameter 50 cm and spins clothes at 1100 rpm (revolutions per minute). How far do a pair of trousers travel if they are spun for 5 minutes? (Assume they stick to the inside of the drum throughout.)
How fast are they going?

Answer: There's a total of 5 × 1100 = 5500 revolutions, each of which is a distance of πd *= 0.5*π *= 1.6 m, so the total distance = 5500 × 1.6 = 8.6 km!*
Speed = distance/time = 8.6/$\frac{1}{12}$ *=*
about 100 kph! (That's why its good if the door won't open until it's finished spinning!)

2.3.14 How many times do the wheels on a car go round when the car travels 1 mile?

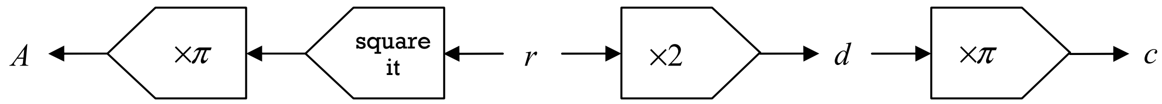
Assume a diameter of about 0.5 m.

Answer:
Circumference = πd *= 0.5*π *= 1.6 m.*
1 mile = 1.6 km, so number of rotations = 1600/1.6 = 1000 times. (This assumes that the wheel doesn't slip at all on the ground.)

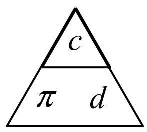
2.3.15 If a car has a turning circle (kerb-to-kerb) of 10 m, estimate the size of the narrowest road in which it could perform a three-point-turn. (Turning circle means that on full lock at low speed the car could just follow a circle of this diameter; i.e., the car could just manage a U-turn in a street 10 m wide.)

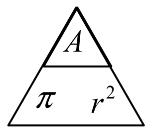
Answer: about half as much, 5 m in this case, because it can turn about 90° clockwise (viewed from above) before reaching the kerb and then reverse another 90° (still clockwise from above) before driving off.
(This assumes that the driver switches from right-lock to left-lock very quickly.)

Converting Area, Radius, Diameter and Circumference of Circles



Using the triangles below, cover up the variable that you want to find, and you can "see" the formula; e.g., $d = \frac{c}{\pi}$, etc.





Fill in the gaps in tables like this (choose where to leave out values). Vary the units.

r	*d*	*c*	*A*
6	12	37.70	113.10
1	2	6.28	3.14
25	50	157.08	1963.50
14	28	87.96	615.75
7.4	14.8	46.50	172.03
11	22	69.12	380.13
28	56	175.93	2463.01
35.8	71.6	224.94	4026.39
254	508	1595.93	202682.99
5	10	31.42	78.54
42	84	263.89	5541.77
0.75	1.5	4.71	1.77

Earth, Sun, Satellites

earth's mean radius: 6.4×10^6 m　　mean distance from earth to sun: 1.5×10^{11} m
height above earth's surface of geostationary satellites: 3.6×10^7 m

Use this data to answer these questions.

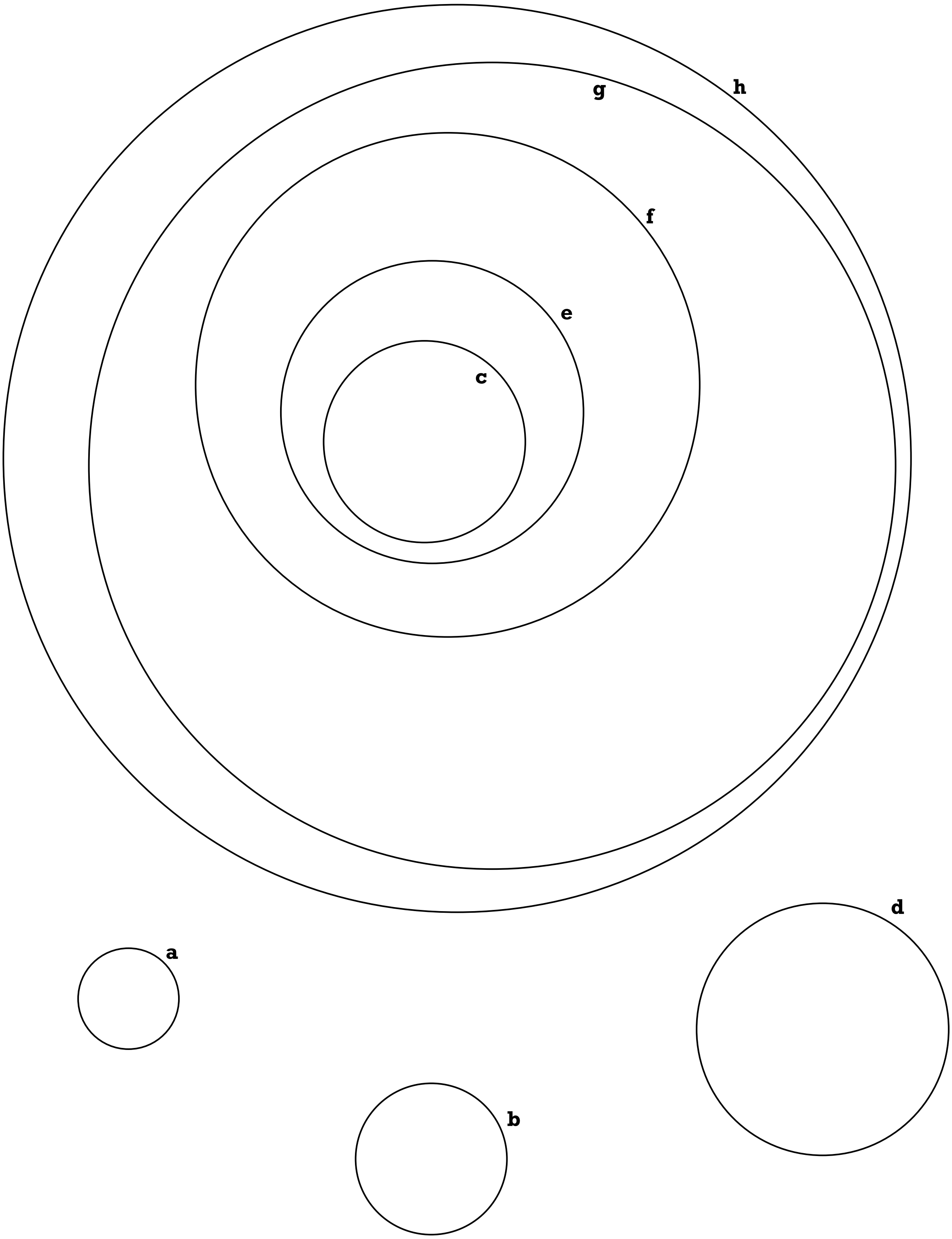
1. How far does someone standing on the equator move in 24 hours?
   First take account of the rotation of the earth.
   Answer: $2\pi r_{earth}$ = 40 000 km.
   Then think about the earth's movement round the sun.
   Answer: $\frac{2\pi r_{earth-to-sun}}{365}$ = 2.6×10^6 km

2. How fast does a geostationary satellite have to move in space?
   A geostationary satellite is one which is always in the same position above the surface of the earth as the earth rotates.
   Answer: $\frac{2\pi r_{earth-to-satellite}}{24}$ = 11 000 kph.

Circles

Measure the *circumference* (use string, a strip of paper or a tape measure) and the *diameter* of each circle.
Record your results in a table.
What do you notice?

Circumference: distance around the edge
Diameter: distance across the middle



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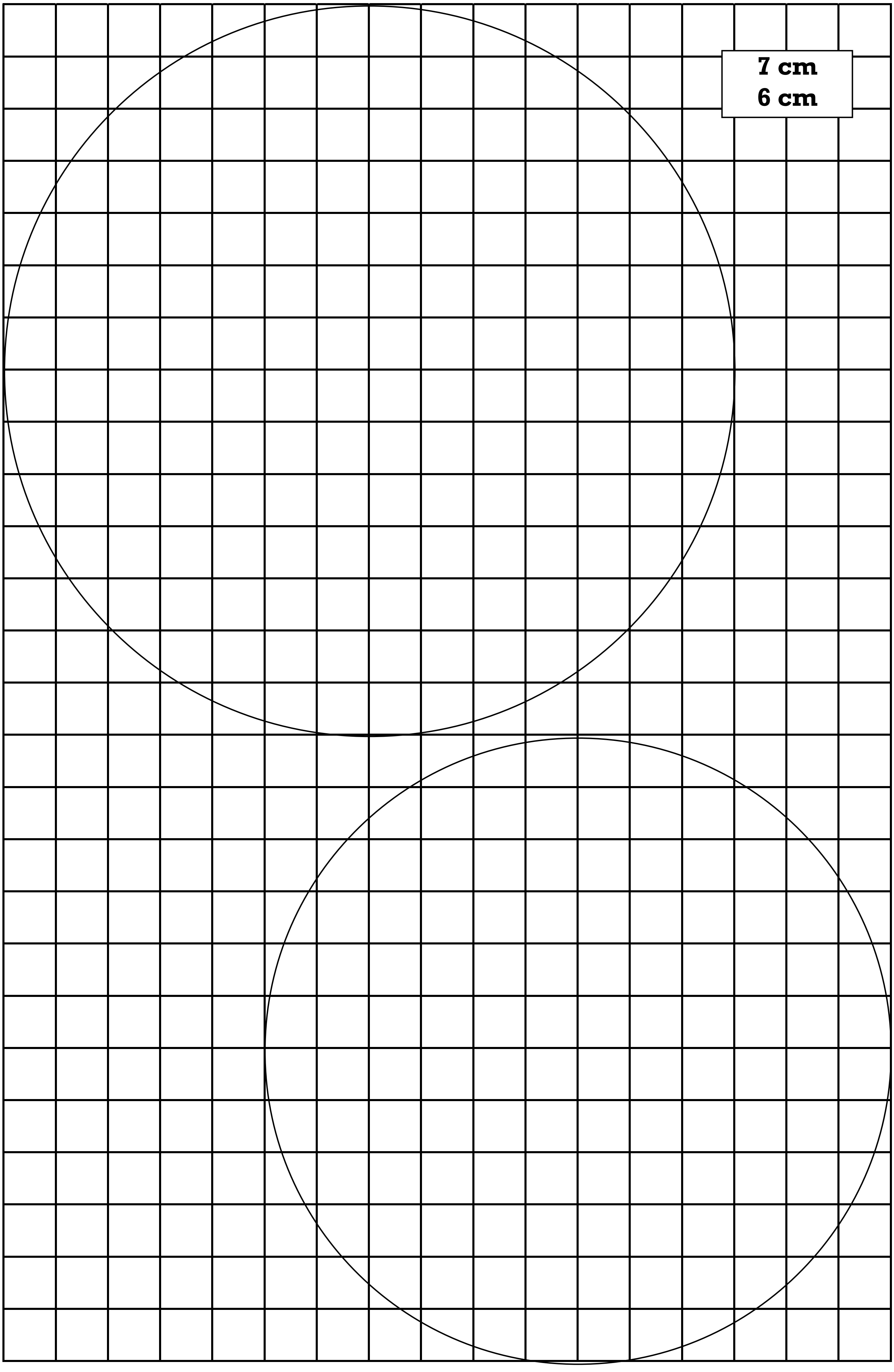
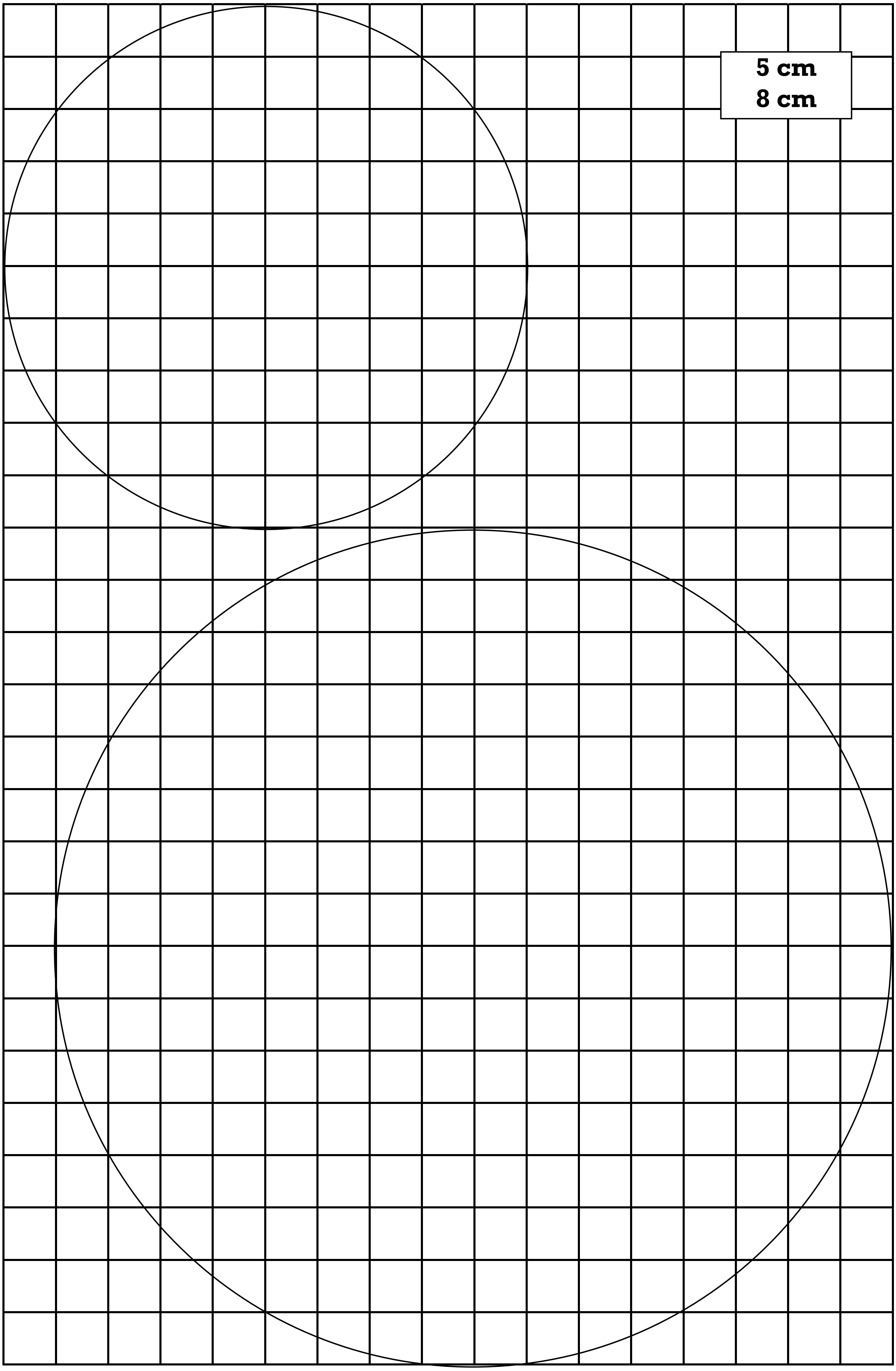

7 cm
6 cm


5 cm
8 cm

2.4 Angles

- "Angle" can sometimes refer to a corner (vertex) or to the size of the angle at that vertex. In diagrams we tend to use capital letters to represent points or sizes of angles and lower case letters to represent lengths of lines; e.g., A , $\hat{A}$, ABC , $A\hat{B}C$, $\angle A$, $\angle ABC$ for angles and a , AB , for lengths.
- Pupils may also need to know the conventional ways of indicating equal angles, equal lengths of lines and parallel lines in diagrams. This is sometimes better covered when dealing with polygons (section 2.1).
- A full-circle protractor (0°-360°) is a lot more convenient than a semicircle (0°-180°) one, and it's an advantage if there is no area missing in the middle and there are continuous lines going out from the centre to the numbers around the edge. Pupils also sometimes find a 360° protractor easier to hold because of the lump at the centre. The reason for the two scales (clockwise and anticlockwise) may need explaining.
- The convention that anticlockwise rotations are positive and clockwise negative is often used.
- Pupils can aim for an accuracy of $\pm 1°$. A sharp pencil helps. Sometimes lines need extending on a drawing to reach the fine scale around the edge of the protractor.
- Angles are revised in sections 2.5 and 2.6.

2.4.1 People maths. Review N, E, S, W ("Naughty Elephants Squirt Water!", or equivalent). Everybody stand up. Let's say this way is north (or work it out or take a vote! **W**est could be the direction of the **w**indows, etc.). Which way is clockwise? Which way is right? Turn 90° clockwise, turn 270° anticlockwise, turn 450° clockwise, etc. Everyone does it at the same time.

Avoid embarrassing pupils who have difficulty distinguishing right and left. "Right is the hand most people write with." The first finger and thumb of the left hand make an "L" shape when held out at 90°.

Establish that for 180° the direction doesn't matter.

Then try it mentally. Sit down. If I said turn 270° anticlockwise and then 90° clockwise which way would you be facing? (N, S, E, W?)

If you started off facing North, you would be facing South.

How much clockwise is equivalent to x ° anticlockwise?

Answer: $(360-x)$°, *if* $0 \le x \le 360$, *or more generally* $(360 - x \bmod 360)$ °.

Give me directions for getting from here to the hall/dining room/head's office, etc.

Not allowed to draw anything or wave arms around. Imagine you were using the telephone.

2.4.2 **NEED** blindfold (clean tea-towels are convenient), prize (e.g., chocolate bar). Choose 2 volunteers, one who doesn't mind being blindfolded and blindfold that one. Escort the blindfolded person to the back of the room and rearrange the desks a little. Place the prize somewhere. The blindfolded person has to get the chocolate bar without touching anything except the floor with any part of him/her. The other volunteer has to give directions but isn't allowed anywhere near the blindfolded person.

Obviously check they can't see anything. Spin round to lose orientation.

On a chair tucked under a table is quite difficult.

You could give them 3 "lives".

Keep close by in case the person falls.

Keep it open-ended and see what they do – generally they'll use left and right, but possibly angles, especially near the end when small movements are necessary.

2.4.3 Covering the special names for angles in particular ranges gives a good opportunity to review > and < or to introduce θ (does anyone know Greek?) as a symbol often used to represent an angle (a bit like x standing for a number in algebra).

The words horizontal *(the same direction as the* horizon*) and* vertical *might need revision.*

These may be ways of adding interest to revision material.

- acute angles: $0 < \theta < 90$;
- right angle (quarter turn): $\theta = 90$;
- obtuse angles: $90 < \theta < 180$;
- straight line (half turn): $\theta = 180$;
- reflex angles: $180 < \theta < 360$;
- full turn: $\theta = 360$.

The "angle facts" that the angles on a straight line add up to 180° and the angles at a point add up to 360° are really just definitions of what we count as "straight" and what we mean by "all the way round" together with our choice of how many degrees to have in a full turn.

Why do you think there are 360° in a full turn?

Just a convention/historical accident; apparently the Babylonians counted in 60's (base 60) instead of 10's like we do.
We could use 2π (radians) or 400 (gradients) or anything you like for a full turn.

We could measure in % (25% for a quarter turn, etc.) or "minutes" (15 min for a quarter turn).

2.4.4 Testing Angle Accuracy.
Draw axes from 0 to 10 horizontally and 0 to 30 vertically. Must use the same scale (e.g., 0.5 cm for 1 unit) on both axes.
As accurately as possible, join (0,0) to (10,10). Measure the angle made by this line and the horizontal axis.

Pupils may enjoy the chance to press the mysterious tan *button on the calculator before they learn about it in trigonometry.*

Answer: Should obviously be 45° because we're bisecting the 90° between the axes, but the calculator knows what it should be: make sure you're in degrees mode and do $\tan^{-1} 1$.

Now join (0,0) to (10,20) and again measure the angle this line makes with the horizontal axis.

Check with $\tan^{-1} 2$ = 63.43494882…°.

Finally (0,0) to (10,30).

Check with $\tan^{-1} 3$ = 71.56505118…°.

Pupils may suspect that the line joining, say, (0,0) to (10,25) would make an angle of $\tan^{-1} 2.5$ with the x-axis.

Further points may be chosen if more practice is needed; this saves photocopying sheets of random angles for pupils to measure, and reviews co-ordinates at the same time. It may also be more interesting.

2.4.5 What is an angle?
(Imagine your little brother/sister wanted to know what your maths homework was about.)

It's quite hard to explain; you can't really say "distance between", etc.
Need to talk about rotating or turning: "a way of saying how much something has turned".

Tell me a job in which you'd have to think about angles?

flying a plane, footballer, plumber, roofer, TV aerial fitter, maths teacher, etc.

2.4.6 Parallel and perpendicular logically crop up here, because parallel lines are straight lines going in the same direction (angle between them = 0°) and perpendicular lines are lines that are at 90° to each other. (Even if the lines go on for ever in both directions, perpendicular lines don't necessarily touch in 3 dimensions. Non-parallel non-intersecting lines are called *skew* lines.)

It isn't enough to say that parallel lines are just lines the same distance apart because so are the circumferences of concentric circles or railway tracks going round a bend. The lines also have to be straight.

Could discuss infinity. Do parallel lines ever meet? Not if they're always a certain (non-zero) distance apart. On a globe "parallel" lines of longitude meet at the N and S poles.

Pupils may know from reflection and refraction in science that "normal" means at 90° (the angles of light rays are measured from lines normal to the surfaces).

2.4.7 Find out how much the Leaning Tower of Pisa leans (could do this for homework).

Answer: about 10°, although it varies year by year as it leans more and then engineers try to straighten it a little.

2.4.8 **NEED** "Which Angles are Equal?" sheets, pencil crayons, protractors perhaps.

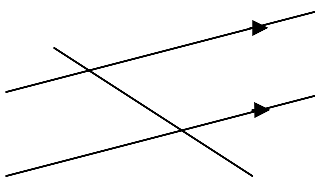
This seems to work better than asking pupils to draw their own lines, because unless you're careful there isn't room to mark the angles clearly. Have some spare copies because when you've gone very wrong it's hard to rescue!

The intention is that pupils draw a circle (about 1 cm radius) at each of the 15 crossing points and then using two colours shade vertically opposite angles the same colour. Equal angles at different crossing points should also be coloured the same colour.

How many colours will you need?

Answer: 6, because there are 3 pairs of parallel lines, and each can intersect any of the others – and $^3C_2 = 3$. Each intersection creates 2 different-sized angles, so altogether there will be 6 different angles.

2.4.9 Angles associated with parallel lines.
Learning the names can be tedious.
First concentrate on identifying which angles are equal to each other and which pairs of angles sum to 180° (supplementary angles).
If you have a rectangular whiteboard and a metre stick (or pole for opening the windows) you can easily lay the stick across parallel sides of the board at different angles to show what equals what; otherwise use a noticeboard or desk.



The line crossing a pair of parallel lines is called a transversal.

Vertically opposite *angles (like scissors or a pair of pupils' rulers, like letter X) – nothing to do with a "vertical" direction, but angles opposite at a "vertex" (= point).*
Corresponding *angles (in corresponding positions, like Chris and Katie, both at the end of a row in the classroom, like letter F).*
Alternate *angles (opposite sides of the line that goes through the parallel pair of lines, like letter Z).*

Interior *angles – the odd ones out because they're not equal but they sum to 180° (like letter C).*

(Unfortunately "C-angles" are "interior" and the "C" doesn't stand for "corresponding".)

Interior angles in polygons *are just the inside angle at each vertex, and they sum to different amounts depending on how many sides the polygon has.*

2.4.10 Clock Angles.
If you look at the minute and hour hands of a clock, there are two angles between them.
I'm only interested in the smaller of the two angles.

Answers:
This smaller angle may be acute, right-angled or obtuse but never reflex.

Tell me a time when the angle is 90°.

3.00 and 9.00. Other times are probably not exact.

Why is 12.15 not exactly correct?

The hour hand will have moved a bit as well.

Will 12.15 be more or less than 90°?

Less.

Tell me another time when the angle will be just less than 90°?

Less: 1.20, 2.25, 3.30, 4.35, 5.40, 6.45, etc.

When will it be just more?

More: 4.05, 5.10, 6.15, 7.20, 8.25, 9.30, etc.

What will the angle be at 9.30?

Hour hand will be half way between 9 and 10 (15° above the 9) so the angle is 90 + 15 = 105°.

Work out the (smaller) angle between the hands at

1. 3.00	**2.** 11.00	**3.** 11.30
4. 3.30	**5.** 12.05	**6.** 4.45

Answers:

1. 90°	**2.** 30°	**3.** 165°
4. 75°	**5.** 27.5°	**6.** 127.5°

(Pupils can do some of this mentally.)

Make use of symmetry (3.30 will be the same angle as 8.30 – not 9.30 – because it's just a reflection in a mirror).

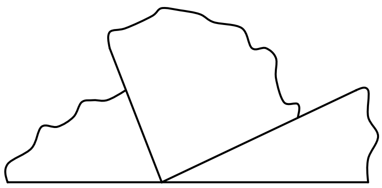
2.4.11 The sum of the three interior angles in a triangle = 180°.

NEED small paper triangles, one per pupil. Hand them out. Who's got a nearly right-angled triangle? Who's got a nearly equilateral triangle?, etc. Hold it up.
Class experiment – between us we're trying all sorts of triangles.

*Be sure that pupils don't think you're saying "***some** *of the angles in a triangle are 180° – and some aren't"!*

You can make these quickly and easily using two or three sheets of coloured A4 paper and a guillotine. Make sure that there are a variety of acute-angled, obtuse-angled and right-angled triangles.

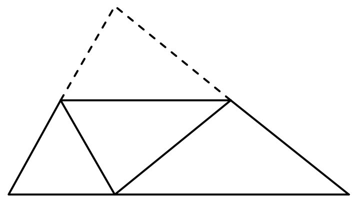
Option 1: You can tear off the corners and arrange the pieces next to one another to make a straight line.



Tearing is better than cutting because it's easy to see which corner is the angle *that was previously in the triangle.*

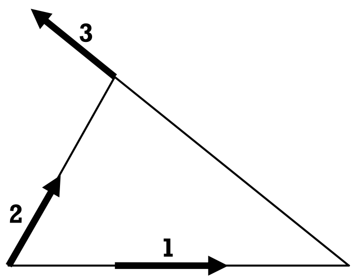
Option 2: Put the longest side horizontal (largest angle at the top) if you can tell, and measure the lengths of the two shorter sides; divide those lengths by two to find the mid-points of those two shorter sides and mark them. Join them with a ruler to give a line parallel to the bottom side and fold the top part down along this line (see diagram right). Fold in the other two corners (vertical fold lines) and all three angles should meet together and make 180°.

Option 2 may be better if many pupils have done option 1 before.



Note that neither option 1 nor option 2 is a proof, *although they can be turned into proofs by thinking about equal angles.*

Dynamic Proof: Draw a "general" triangle on the board and put a large cardboard arrow or similar straight object (must have distinguishable ends) in the middle of the bottom side (see diagram right, **1**).
We're going to use the arrow to "add up" angles.
Slide it to the left corner (no change in direction so no angle yet) and rotate it to "measure" the interior angle at that corner (**2**). Slide it to the top corner, and, still rotating anticlockwise, "add on" that corner angle (actually the angle vertically opposite to it, which is equal) (**3**). Slide it, without changing direction, to the final (right) corner and "add on" that angle (still anticlockwise), leaving the arrow pointing 180° relative to its initial position (**1**).



(This is similar to the standard proof that the exterior angles of any polygon add up to 360° by walking around it.)

Work like this shows the dynamic aspect of angles – they're a measure of turning movement.

2.4.12 How many times in the course of 12 hours are the hands of a clock at right angles to each other?

Pupils can invent similar puzzles.

Answer: 22 times
Imagine starting and finishing at 12.00.
The hands will be at right angles twice in every one hour period, except that we will count 3.00 and 9.00 twice (because they occur on the hour), so we have to subtract those two.
24 – 2 = 22 times.

2.4.13 The hour hand and minute hand on an analogue clock coincide at 12 noon. When is the next time when they coincide exactly?

Hint: It won't be exactly 1.05 pm.
Will it be before or after? *(After)*

If they coincide t *hours after 1 o'clock, then* $\theta_h = 30(t+1)$ *and* $\theta_m = 360t$*, so solving simultaneously gives* $30(t+1) = 360t$ *and so* $t = \frac{1}{11}$ *hour after 1 o'clock.*

Answer: Can use simultaneous equations, even the concept of angular velocity, if you like.

A neater way is to see first that the occasions when the hands coincide will occur regularly. (At the 1.05-ish time, imagine rotating the painted numbers so that it reads 12 noon again – you could carry on like this.)
Since it will happen 11 times in 12 hours, each coincidence will occur after $\frac{12}{11}$ *of an hour; i.e., at 1.05 and 27 secs, 2.10 and 55 secs, etc.*

2.4.14 Triangles. Draw up a table like this (big enough to contain *drawings*):

	scalene	**isosceles**	**equilateral**
acute-angled			
obtuse-angled			
right-angled			

For the top left square, if a triangle can be both scalene and acute-angled, draw an example. Put X if it's impossible, and try to say why. Complete the table.

There are two systems for naming triangles: by their angles or by the lengths of their sides.

Note that "acute-angled" means all the angles are acute, whereas "obtuse-angled" means only that there is one *obtuse angle (more would be impossible –* *see section 2.4.15**).*

Answers:
Equilateral triangles must be acute-angled, because all the angles have to be 60°.
All other combinations are possible.

2.4.15 What sorts of angles can a triangle have?

Here are some more specific questions.
If the answer is yes, make up an example and draw or list the angles; if the answer is no, try to explain why not.

Can a triangle have (as *interior* angles) …
1. an obtuse angle?
2. two obtuse angles?
3. an obtuse angle and a right angle?
4. an obtuse angle and an acute angle?
5. a reflex angle;
6. two right angles;
7. a right angle and an acute angle?
8. four acute angles?

Answers:

1. *yes, lots of examples;*
2. *no, because their total would be >180°;*
3. *no, because their total would be >180°;*
4. *yes, lots of examples;*
5. *no, because it would be >180°;*
6. *no, because their total would be 180° and there needs to be a non-zero third angle;*
7. *yes, lots of examples;*
8. *no, because although their total might be 180° (e.g., 40˚, 45˚, 45˚, 50˚), a "tri-angle" has to have exactly 3 angles!*

Note that we are always talking about the interior *angles of the triangles.*

2.4.16 I'm thinking of a triangle. Tell me how big its angles are and what kind of triangle it is.
I'll call the angles in each one A, B and C.

1. Angle B is twice the size of angle A and angle C is three times the size of angle A;
2. Angle A is twice the size of angle B and angle C is the same size as angle B;
3. Angle B is twice the size of angle A and angle C is three times the size of angle **B** (not same as question 1);
4. Angle B is four times the size of angle A and angle C is five times the size of angle A.

Answers:

1. *A = 30°, B = 60°, C = 90°; scalene right-angled triangle;*
2. *A = 90°, B = 45°, C = 45°; isosceles right-angled triangle;*
3. *A = 20°, B = 40°, C = 120°; scalene obtuse-angled triangle;*
4. *A = 18°, B = 72°, C = 90°; scalene right-angled triangle.*

These can be solved by forming equations or by trial and improvement or by "intuition".

Pupils can make up their own for each other.

2.4.17 Polygon Angles.

("Interior" angles in polygons means something different from "interior" angles formed when a straight line crosses a pair of parallel lines.)

Drawing polygons with various numbers of sides and measuring and summing the interior angles of each tends to give very inaccurate results, although it is a possible approach.

What do the angles inside a square add up to?
What if I just draw any quadrilateral?
(Keep it convex for now.)

Most will know/guess 360°.
Show how it can be split by a diagonal into two triangles. Colour the angles in one triangle red and the other blue. What do the blue angles add up to?, etc.

Use this trick of splitting into triangles to work out the total interior angle in polygons with sides from 5 all the way up to 10. Choose one vertex and joining this to all the other vertices, so dividing the polygon into triangles.
If the polygon happens to be regular, what can you say about each of the interior angles?

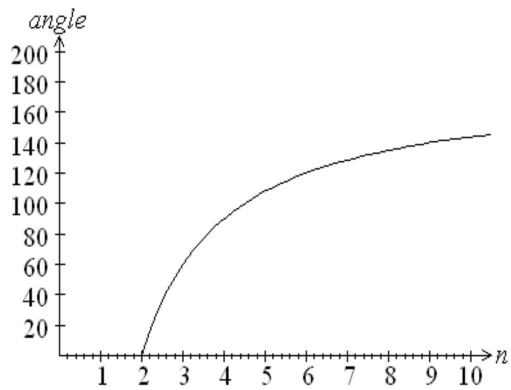
If the polygon has n sides (and so n vertices), this method will divide it into $n-2$ triangles.
So the total interior angle = $180(n-2)$.

In a regular polygon, the interior angles will be equal, so each will be $\frac{180(n-2)}{n} = 180 - \frac{360}{n}$.

no. of sides ***n***	***no. of triangles*** ***n – 2***	***total interior angle***	***size of each angle if polygon is regular***
3	*1*	*180*	*60*
4	*2*	*360*	*90*
5	*3*	*540*	*108*
6	*4*	*720*	*120*
7	*5*	*900*	*128.6*
8	*6*	*1080*	*135*
9	*7*	*1260*	*140*
10	*8*	*1440*	*144*

If the polygon contains one or more reflex angles, this method of dissection into triangles doesn't work.
In that case, one or more points have to be chosen inside the triangle and these joined to as many vertices as possible. In this way, the polygon can always be divided into triangles, and each additional internal "point" contributes an extra 360° to the total angle, and this has to be subtracted.
(The total interior angle for a concave polygon like this is always the same as for a convex polygon with the same number of sides.)

Plot a graph of "size of one interior angle of a regular n-gon" against n.
Make a prediction for $n = 100$.
Why does this happen?



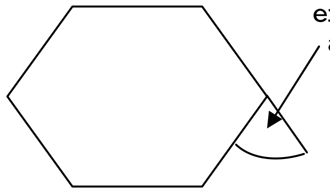
The graph gets closer and closer to 180° as the number of sides the polygon has goes up.
(There is an asymptote at 180°.)
When $n = 100$, angle = 176.4°.
As the number of sides goes up, the polygon looks more and more like a circle, and if you zoom in on any part of the circumference of a circle it looks almost like a straight line (180° interior angle).

Common-sense says the angle at each corner will get bigger as the polygon gets more sides.

Mathematically, if θ is the angle, then $\theta = \frac{180(n-2)}{n} = 180 - \frac{360}{n}$, and as n gets larger and larger, $\frac{360}{n}$ gets smaller and smaller, so that θ increases, getting closer and closer to 180°.
(As $n \to \infty$, $\theta \to 0$.)

An alternative way to find the interior angle of a regular polygon is to begin with the exterior *angles (which sum to 360° for any polygon), so in a regular polygon each must be $\frac{360}{n}$.*
Therefore, since each exterior angle and interior angle together must make a straight line, each interior angle must be equal to $180 - \frac{360}{n} = \frac{180(n-2)}{n}$.

2.4.18 Exterior angles.
NEED newspaper and scissors.
The "exterior angle" doesn't mean all the angle outside the shape at each vertex; it means the *change in direction* at each vertex, so that exterior angle + interior angle at each vertex equals 180° (see diagram right).
(For a concave polygon, where the interior angle at one or more vertices is a reflex angle, we say that there is no exterior angle at that vertex.)
Place sheets of newspaper on the floor and imagine walking round it (start in the middle of one of the sides). The total change in direction is 4 × 90° = 360°. This will always happen. (Cut the paper so that it makes a more unusual polygon and do it again.) After we've walked round the whole shape we're always pointing back in the same direction.



(Or you can go outdoors and use or draw with chalk a shape on the ground.)

For this to work with a concave polygon we need to count the exterior angle where there is a reflex interior angle as negative, *because at that corner we change from going clockwise to going anticlockwise (or vice versa).*

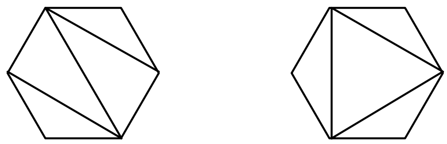
With an equilateral triangle, the exterior angles are 120°. Exterior angles are important when doing LOGO programming (see section 3.8).

2.4.19 Any convex polygon with n sides can be split up into $n-2$ triangles. How many ways are there of doing that?

They fit the formula

number of ways = $\dfrac{{}^{2n-4}C_{n-2}}{n-1}$.

For instance, for $n=6$ *you can have, e.g.,*



Answer:

n	***no. of ways***	*n*	***no. of ways***
3	*1*	*8*	*132*
4	*2*	*9*	*429*
5	*5*	*10*	*1430*
6	*14*	*11*	*4862*
7	*42*	*12*	*16796*

These are the Catalan *numbers (Eugène Catalan, 1814-1894).*

It might seem that for $n \geq 5$ *there will just be* n *ways, because* $n-3$ *vertices can be joined to each of the* n *vertices, but there are more than this because the cuts need not all meet at one point. For* $n \geq 6$ *there are dissections like those on the left.*

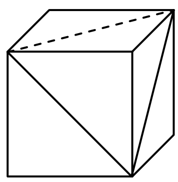
2.4.20 Constructing and solving equations from polygon and parallel line angles;
e.g., a triangle has angles $2x$, $3x$ and $5x$; how much are they?

Answer: $10x = 180$*, so* $x = 18$ *°, so the three angles are 36°, 54° and 90°.*

It's easy to make up this kind of thing or find examples in books.

2.4.21 Angles in 3 dimensions.
Two diagonals are drawn on two adjacent faces of a cube. If they meet at a vertex, what angle do they make with each other?

Answer: need to do a diagram or make a model.



If you add in a third diagonal, you can see that you get an equilateral triangle (all sides are the same length), so the interior angles are all 60°.

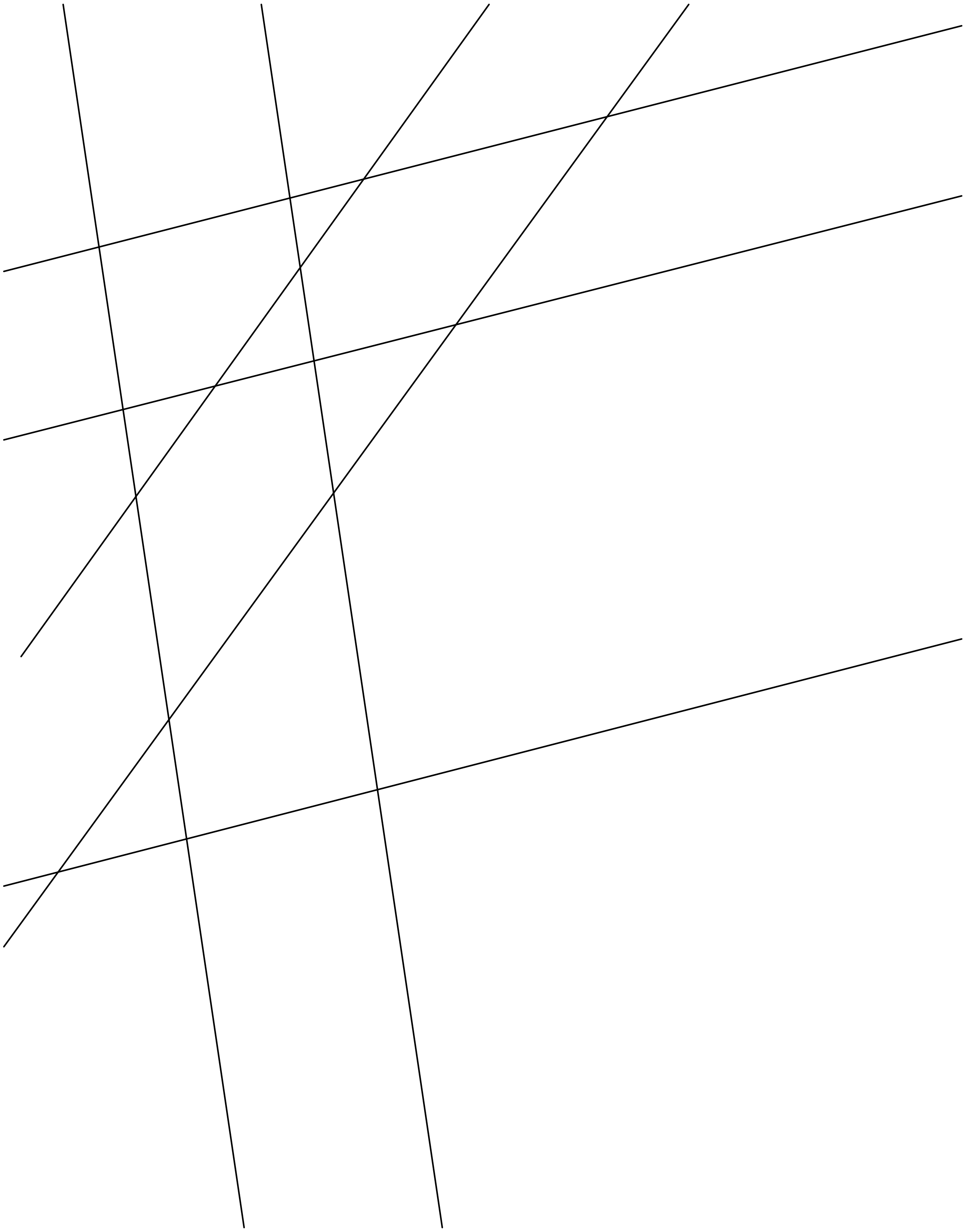
People often go for complicated methods like trigonometry or vectors, but particularly in geometry there is often an easier and more elegant solution.

2.4.22 How much can an object lean without toppling? For instance, a 3 × 1 × 1 cuboid brick standing on its end – what's the steepest slope it can balance on?
For this brick,
maximum angle of slope = $\tan^{-1}\frac{1}{3} = 18.4$ *°.*

Answer: If the object is uniform (the same all the way through), then its "centre of mass" will lie at the geometric centre. The object will be stable if a vertical line going through the centre of mass passes within the base.
(We assume that there is plenty of friction so the block won't slide down the slope.)

Which Angles are Equal?

- Use arrows to show which lines below are parallel to each other.
- Every time two lines cross each other, they create four angles.
  There are 15 crossing points in the drawing below, so there are 60 angles.
  Use colour to show which angles are the same size as each other.



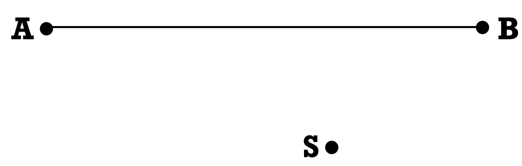
2.5 Bearings

- There are three rules of bearings: always measure from north;
  always measure clockwise;
  always give 3 digits.
  (If there's someone in the class who does orienteering, you could ask him/her (in advance) to bring a compass and teach the class how bearings work.)
  Sometimes you have to draw in a north line at the point you're measuring the bearing *from*.
- A full-circle protractor (0°-360°) is a lot more convenient than a semicircle (0°-180°) one, and it's an advantage if there is no area missing in the middle and there are continuous lines going out from the centre to the numbers around the edge. Pupils also sometimes find a 360° protractor easier to hold because of the lump at the centre. The reason for the two scales (clockwise and anticlockwise) may need explaining.
- Pupils can aim for an accuracy of ± 1° and ± 1 mm. A sharp pencil helps. Sometimes lines need extending on a drawing to reach the fine scale around the edge of the protractor.
- Local maps are usually much more interesting than remote or invented ones in textbooks. (You could ask the Geography department if you can borrow some local maps.)

2.5.1 **NEED** A4 plain paper.
Coastguard Stations.
Facing the class, pick a pupil at your front left. "Beth is a coastguard station. Ashley (front right) is another coastguard station. And Sally (somewhere in the middle of the room) is a sinking ship!"
On the board, begin a drawing (see right).

Fits nicely on A4 paper.

(The drawing below is the right way round for the class.)



Sally radios in to Ashley and to Beth to say that she's sinking – send help! But her GPS is broken and she doesn't know where she is. And by the way it's night time and foggy.

(GPS = Global Positioning System – a very accurate way of telling where you are that uses satellites.)

A and B measure the directions they receive Sally's message from.

	from A	**from B**
Ship	157°	213°

The coastguard stations are 16 km apart, so draw a line at the top of the page (portrait orientation) 16 cm long (the scale is 1 cm to 1 km). We'll take north as up the page.
Do a scale drawing to find out how far away S is from A and B.

This is called triangulation.

A helicopter radios in to go and rescue S.

	from A	**from B**
Helicopter	103°	244°

Find where the helicopter is and work out how far the helicopter has to go and on what bearing to pick up Sally.

Answers:

	from A	**from B**
Ship	16.2 km	17.8 km
Helicopter	11.1 km	5.7 km

Message to helicopter: "Travel 13.2 km on a bearing of 200° to intercept the ship."

(Although this scenario is not completely realistic, pupils generally realise that but enjoy it anyway.)

(All of this assumes that the ship is stationary.)

2.5.2 **NEED** "Crack the code" sheets, protractors.

Pupils can make up their own messages for each other using the same code.

("Remember I know the code so don't say anything you wouldn't want me to read!")

Answer:
The message is "Always give three digits!"
The key is as follows.

A	243	**B**	016	**C**	260	**D**	314
E	148	**F**	234	**G**	306	**H**	072
I	209	**J**	167	**K**	096	**L**	056
M	036	**N**	124	**O**	105	**P**	278
Q	224	**R**	029	**S**	183	**T**	332
U	251	**V**	291	**W**	346	**X**	196
Y	269	**Z**	133				

2.5.3 What is the connection between the bearing of A from B and the bearing of B from A?
Can you write a rule that works for any bearing ≥ 000° and < 360°?

Answer:
If the bearing of A from B is θ*, then the bearing of B from A is*
$\theta + 180$ *if* $0 \le \theta < 180$*, and*
$\theta - 180$ *if* $180 \le \theta < 360$*.*
Or you can say $(\theta + 180) \bmod 360$*.*

2.5.4 How could this happen?
I take off in my aeroplane and head South. I turn 90° to the left and head East. After a while, I turn 90° to the left again and head North, and without changing direction again, I land back where I started.

Euclid (about 330-270 BC) wrote "Elements".

The sum of the angles in a triangle on a sphere (a "spherical triangle") comes to more than 180°, more the larger the triangle. Very small triangles behave more or less Euclidean.
What kind of surface would have the sum of the angles in a triangle <180°?

Triangles behave "normally" on the surface of a cylinder. You can unroll a surface like this into a flat sheet; it has no "intrinsic curvature".

Answer: I must have started and finished at the North pole; the middle leg of the journey was along the equator.

A globe or someone's football makes this clear. When you think about angles on a sphere, you're doing non-Euclidean geometry (although all the North lines are going in the same direction, they meet – at the North pole).

(People sometimes think that the answer is the inside *of a sphere, because it curves* away *from you instead of* towards *you, but imagine that the sphere was made of glass – it's actually the same triangle on both sides.)*
The answer is to do with hyperbolic geometry – you need a surface with "negative curvature"; e.g., a bicycle saddle.

2.5.5 **NEED** maps of the local area; an A4 portion should be enough (you could ask the Geography department if there are some you could borrow).
Find the school, draw in a North line there, and work out the bearings off all the major places from the school.
Record the values in a table (place in one column, bearing in another).

Depending on how wide your catchment area is, you may want about 3.5 inches to a mile or 1.25 inches to a mile so as to include most of the places where pupils live or go.

2.5.6 Treasure Map.
Make one, perhaps for an island, mark on the starting point and mark faintly in pencil an X where the treasure is. Draw on dangers (swamps, man-eating tigers, sharks, dangerous rocks, cliffs, etc.), but make sure there's a safe route from the starting point to the treasure. Decide which direction is North, mark that on, and put a scale (say 1 cm = 1 m).
On a separate sheet list instructions using bearings for getting safely to the treasure; e.g., "Go 5 m on a bearing of 045°, then go …".
Rub out the X thoroughly afterwards!

If pupils are likely to spend a long time drawing the island and its hazards and not much time working out bearings, you could insist that they mark the safe route as they draw in the dangers and work out the bearings and instructions as they go along, rather than leaving that until the end!

Pupils could do the route on a second (thin) piece of paper laid on top and remove it at the end.

See if someone else can follow your route.

2.5.7 What methods are there for finding where North is? What are the advantages and disadvantages of each?

Answers: (some possibilities)

- *compass (simple, but only works if you have one handy and aren't too near anything with a strong magnetic field);*
- *use a map (or local knowledge) and a landmark;*
- *point the minute hand of an analogue watch at the sun, and bisect the angle between it and the hour hand (don't need special equipment, but only works if you can see the sun);*
- *find the North Star (need to know how to locate it, have to be able to see the stars);*
- *global positioning system (very accurate, but obviously you need to have one, and this was no good before they were invented!).*

2.5.8 **NEED** "Fancy Fields" sheets.
An opportunity to review polygon names and scale drawing.

Answers:

A	*rectangle*	***B***	*equil. triangle*
C	*regular hexagon*	***D***	*square*
E	*isos. r-angled tri*	***F***	*isos. trapezium*
G	*parallelogram*	***H***	*regular octagon*

2.5.9 Plot these points and work out the bearings of the *first* from the *second*.
Take the y -axis as North.

1. (2,3) from (0,0)
2. (4,4) from (0,0)
3. (–2,5) from (0,0)
4. (6,–1) from (0,0)
5. (–2,–1) from (0,0)
6. (3,0) from (1,1)
7. (3,1) from (5,2)
8. (–4,–1) from (0,1)
9. (–5,3) from (–2,0)
10. (3,–5) from (–2,–1)

This could be done by measuring accurately or by calculating using trigonometry and angle facts.

Answers:

1	*034°*	***2***	*045°*
3	*338°*	***4***	*099°*
5	*243°*	***6***	*117°*
7	*243°*	***8***	*243°*
9	*315°*	***10***	*129°*

Why are the answers to three of the questions the same?

The lines in questions 5, 7 and 8 all have the same gradient (0.5) and direction (they're the same vectors*).*

2.5.10 Which way does an easterly wind go?

On a map, "northings" run east-west and "eastings" run north-south.

*Answer: An "easterly wind" usually means **from** the east; i.e., heading west, whereas an "easterly current" goes **to** the east (from the west).*

For this reason, it's generally clearer to describe the direction it's coming from or going to explicitly.

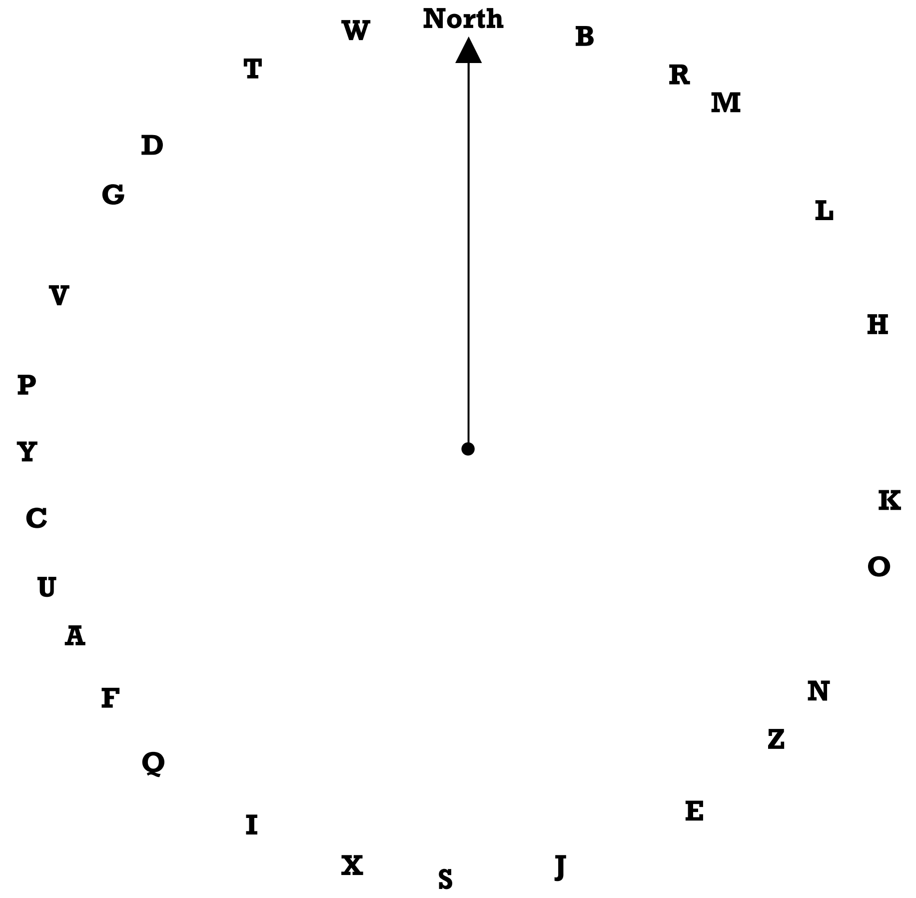
2.5.11 In some religions, people face in a particular direction to pray. Find out more about this.

This would theoretically be impossible if you were either at Jerusalem/Mecca or on the exact opposite side of the world from there (on a ship in the middle of the Pacific Ocean!).

Answer: Some Jewish, Christian and Muslim groups pray facing East. This may have had something to do with the sun rising in the East, or the Garden of Eden being planted "in the East" (Genesis 2:8) or with many believers living to the west of the "Holy Land". Some Jews face Jerusalem from wherever they are along the direction of a Great Circle, and Muslims face Mecca by a similar method.

Crack the Code (Bearings)

Follow the bearings to find the letters.
What is the message?



243° 056° 346° 243° 269° 183° 306° 209° 291° 148° 332°

072° 029° 148° 148° 314° 209° 306° 209° 332° 183°

Crack the Code (Bearings)

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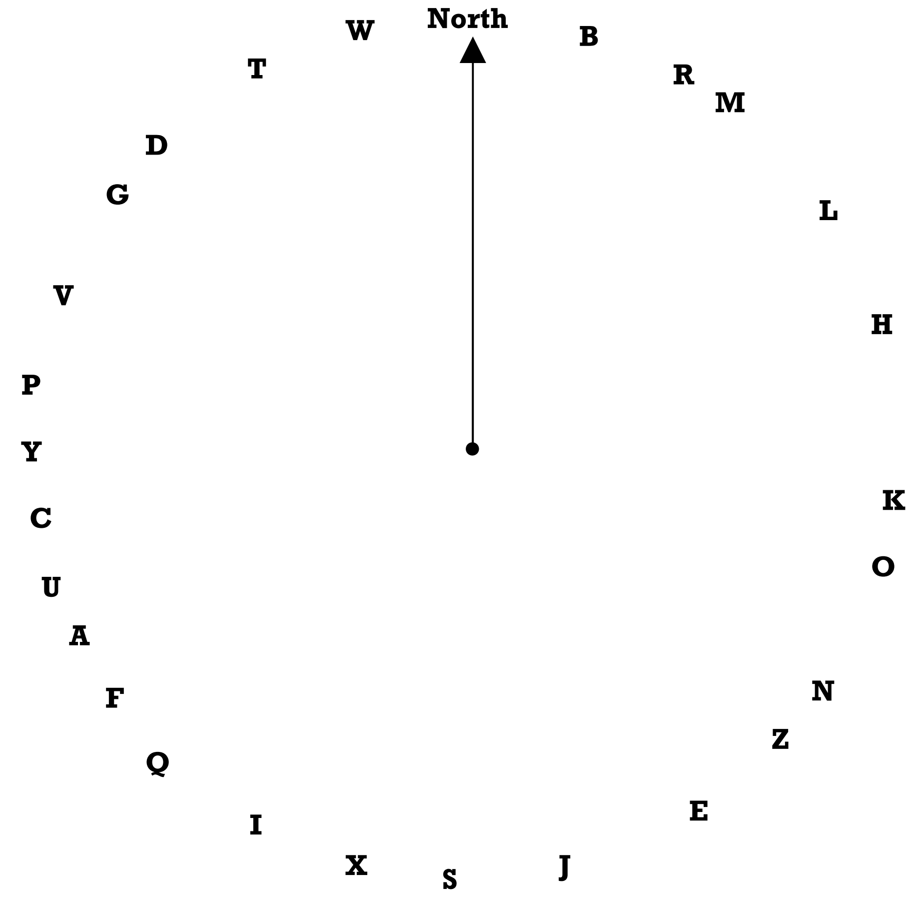


243° 056° 346° 243° 269° 183° 306° 209° 291° 148° 332°

072° 029° 148° 148° 314° 209° 306° 209° 332° 183°

Fancy Fields

I have eight interesting-shaped fields on my land.
I have walked around the perimeter of each of them recording the bearings and distances for each of the sides.
Look at my data and try to say what shape each field is.
Then do an accurate drawing for each one to see if you are right.
Use a scale of 1 cm representing 100 m.

Field A
A bearing of 000° for 400 m;
then a bearing of 090° for 800 m;
then a bearing of 180° for 400 m;
and finally a bearing of 270° for 800 m.

Field B
A bearing of 000° for 400 m;
then a bearing of 120° for 400 m;
and finally a bearing of 240° for 400 m.

Field C
A bearing of 090° for 500 m;
then a bearing of 150° for 500 m;
then a bearing of 210° for 500 m;
then a bearing of 270° for 500 m;
then a bearing of 330° for 500 m;
and finally a bearing of 030° for 500 m.

Field D
A bearing of 045° for 600 m;
then a bearing of 135° for 600 m;
then a bearing of 225° for 600 m;
and finally a bearing of 315° for 600 m.

Field E
A bearing of 000° for 400 m;
then a bearing of 135° for 566 m;
and finally a bearing of 270° for 400 m.

Field F
A bearing of 037° for 500 m;
then a bearing of 090° for 500 m;
then a bearing of 143° for 500 m;
and finally a bearing of 270° for 1100 m.

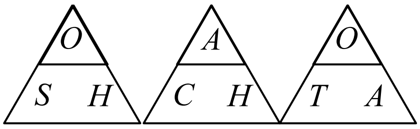
Field G
A bearing of 090° for 600 m;
then a bearing of 220° for 400 m;
then a bearing of 270° for 600 m;
and finally a bearing of 040° for 400 m.

Field H
A bearing of 135° for 400 m;
then a bearing of 090° for 400 m;
then a bearing of 045° for 400 m;
then a bearing of 000° for 400 m;
then a bearing of 315° for 400 m;
then a bearing of 270° for 400 m;
then a bearing of 225° for 400 m;
and finally a bearing of 180° for 400 m.

Extra Task Design some fields of your own and work out instructions (using bearings and distance) for placing the fence around the edge.

2.6 Trigonometry

- A topic that builds heavily on many others: ratio, similarity and enlargement, angles and lengths, calculator use, rearranging formulas, Pythagoras' Theorem, rounding to dp or sf.
- SOHCAHTOA is a common and not too difficult mnemonic. But some people prefer a whole sentence; e.g., "*S*everal *O*ld *H*orses *C*art *A*way *H*appily *T*onnes *O*f *A*pples." Pupils may remember them better if they invent their own and make them funny.
  A convenient way for using SOHCAHTOA is as three formula triangles (below); these can be written at the top of a page of trigonometry work. Each formula triangle has one of A , O or H missing, so we ask "What *aren't* we interested in?" Let's say we don't know the adjacent side and we don't need to work it out, so we want the formula without A in it; therefore $\sin$, the first triangle formula.
  Then decide whether we need the opposite side, the hypotenuse or the angle, and write down a formula for opp , hyp or $\sin x$ from the first triangle formula.



These work by covering over the variable you want to reveal the formula for it.
(If you cover S , you can see $\sin x = \frac{opp}{hyp}$; cover H and you can see $hyp = \frac{opp}{\sin x}$, and cover O and you can see S and H next to each other, so $opp = \sin x \times hyp$. Similarly for the other formula triangles.)

- Pupils can get into a habit of putting all the information onto a clear drawing and labelling the three sides with hyp , opp and adj before they start any calculations.
- Remind pupils to have calculators in the correct mode (degrees or radians). Sometimes the simplest way to sort out a messed-up calculator is to use a biro to press the "reset" button on the back.

2.6.1 **NEED** "Trigonometry Investigation" sheets.
Clearly all the triangles are similar (enlargements of one another), and any ratio of corresponding sides like this should be equal.

This task takes advantage of the fact that $\tan 35$ *is very close to 0.7 (actually 0.7002075…).*

Approximate results:

	opp (mm)	***adj (mm)***	***opp/adj (2 dp)***
a	*35*	*50*	*0.70*
b	*63*	*90*	*0.70*
c	*73*	*105*	*0.70*
d	*46*	*65*	*0.71*
e	*32*	*45*	*0.71*

You can also measure hyp and calculate the $\sin$ and $\cos$ ratios.
($\sin 35 = 0.57$ and $\cos 35 = 0.82$, each to 2 dp.)

2.6.2 You could begin the topic by investigating the "mysterious" $\sin$, $\cos$ and $\tan$ buttons on the calculator.
They are functions that convert an angle in degrees (make sure they're in degrees mode) into a number.
What's the biggest number you can make $\sin$ give you?
What's the smallest?

1, –1 ($\sin$ *values are always between 1 and –1).*

What's it got to do with a right-angled triangle that has that size angle as one of its angles?

Pupils can experiment with the functions to see what happens.

2.6.3 Plot graphs of $\sin x$, $\cos x$ and $\tan x$ against x and discuss their properties.
Plot values for $0° \le \theta \le 720°$, say.
What do you notice?

Key values of the functions at 0°, 30°, 45°, 90°, etc. can be noted (see sheet).

You could discuss the fact that sinusoidal graphs turn up in science; e.g., alternating current/voltage; electric/magnetic fields in electromagnetic waves (e.g., light); the sound wave of a tuning fork is nearly sinusoidal.

Fourier Analysis *(Jean Baptiste Fourier, 1768-1830) is a way of building up a waveform by adding together sinusoidal waves of different amplitudes and frequencies.*

Things to spot include these:

- $\sin x$ *and* $\cos x$ *are both sinusoidal in shape and periodic functions with the same period of 360°;* $\cos x$ *is just 90° ahead of* $\sin x$ *(called a "phase shift").*
- *They both take values between – 1 and 1 ("amplitude" = 1).*
- $\tan x$ *is different in having a period of only 180° and in taking all values from* $-\infty$ *to* ∞. *It is also a discontinuous graph, with vertical asymptotes at* $\pm 90°$, $\pm 270°$, *etc.*
- $\cos x$ *is an* even *function, symmetrical about the vertical axis (* $\cos(-x) = \cos x$ *), whereas* $\sin x$ *and* $\tan x$ *are* odd *functions, having rotational symmetry of order 2 about the origin (* $\sin(-x) = -\sin x$ *and* $\tan(-x) = -\tan x$ *).*

2.6.4 Snell's Law in Science (Willebrord Snell, 1580-1626).
When a beam of light travels from one medium to another, the angle of refraction r (in the second medium) is related to the angle of incidence i (in the first medium) by Snell's Law, $\sin r = \frac{\sin i}{n}$, where the constant of proportionality n is the refractive index.

Plot r against i for a couple of different values of n; e.g., $n = 1.5$ (for light going from air into glass) and $n = \frac{1}{1.5} = \frac{2}{3}$ (for light going from glass into air).

For air and water the values are ${}_{air}n_{water} = \frac{4}{3}$ (air to water) and ${}_{water}n_{air} = \frac{3}{4}$.

Answers:
Angles are measured from the normal (a line perpendicular to the boundary).

For $n = 1.5$, r *increases less quickly than* i, *and when* i *reaches its maximum value of 90°,* $r = \sin^{-1}(\frac{1}{n}) = 41.8°$, *the so-called "critical angle" for glass-air.*
For $n = \frac{2}{3}$, r *increases more quickly than* i, *and this time when* i *reaches the critical value of 41.8°,* i *no longer has any value (because* $\sin$ *can never be >1). This corresponds to the light beam not being refracted at all: "total internal reflection" takes place.*

For water-air, the critical angle is 48.6°.

2.6.5 Converting gradients into angles.
We already know that the line $y = x$ (gradient of 1) makes an angle of 45° with the x-axis.
What is the angle that the line $y = 2x$ makes?

Answers:
$y = 2x$ *makes an angle of* $\tan^{-1} 2$ *= 63.4°.*
$y = mx$ *makes an angle of* $\tan^{-1} m$, *and if you count a negative angle as a clockwise turn, then this works even when* m *is negative.*

2.6.6 **NEED** Old-fashioned trigonometrical tables.
You could explain how hard life was in the old days (!) or how clever calculators are today being able to work out $\sin$, $\cos$ and $\tan$ so quickly. Pupils could try to find out for homework how the calculator does it.

Pupils can try this out for themselves on a spreadsheet. Can you beat the calculator by finding the next decimal place after the last one the calculator shows?

Answer:
They use power series such as
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$ *in radians.*
Taking enough terms, you can get an answer as accurately as you like. (You first have to convert the angle from degrees.)

Spreadsheets can handle more decimal places than most calculators.

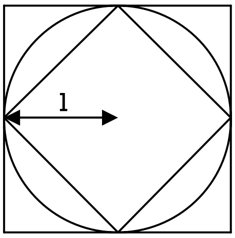
2.6.7 Assuming that the Leaning Tower of Pisa leans at an angle of 10° and is 55 m high, how far does the top lean out over the bottom?

Answer: $55 \tan 10 = 9.7$ *m.*

2.6.8 Approximating π from regular polygons.
Two approaches (see below and right):

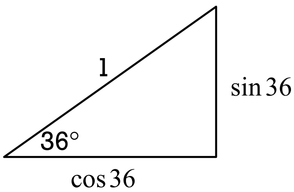
1. Area:

π is the area of a unit circle (a circle of radius 1) and we can approximate that area by finding regular polygons that just fit into the unit circle and just fit outside the unit circle and working out their areas.
Start with squares.

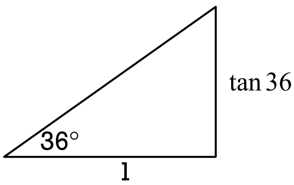


The outer square has area 2 × 2 = 4 sq units.
The inner square has area $4\times\frac{1}{2}\times1\times1=2$ sq units (by dividing it up into four triangles).
So this gives $2<\pi<4$ (not a very good approximation, but it's a start!).

Next try regular pentagons, and so on.
The pentagons can be divided into 5 congruent isosceles triangles by joining each vertex to the centre of the circle. Each of these triangles can be cut in half to give two congruent right-angled triangles.
For the pentagon *inside* the circle, we have



so total area = $10\times\frac{1}{2}\cos36\sin36$ = $5\cos36\sin36$.
For the pentagon *outside* the circle, we have



so total area = $10\times\frac{1}{2}\tan36=5\tan36$.
So now $5\sin36\cos36<\pi<5\tan36$, giving 2.38< π <3.63, a better approximation.

In general, for an n-sided polygon we obtain $n\sin\left(\frac{180}{n}\right)\cos\left(\frac{180}{n}\right)<\pi<n\tan\left(\frac{180}{n}\right)$, so putting in n = 100 gives 3.1395< π <3.1426, and taking n = 100 000 gives π as accurately as can be displayed on a calculator.

2. Perimeter and Circumference:

2π is the circumference of a unit circle, so we can approximate that length by finding the perimeters of regular polygons that just fit into and outside the unit circle; i.e., the same regular polygons as used on the left.

Here, the outer square has perimeter 4 × 2 = 8 units and the inner square has perimeter 4 × $\sqrt{2}$ = 5.66 units, using Pythagoras' Theorem.
So $5.66<2\pi<8$, so $2.83<\pi<4$, giving a different (slightly better) approximation from that on the left.

Using the same drawing as on the left, we have total perimeter of the pentagon *inside* the circle = $5\times2\sin36$ = $10\sin36$.

And total perimeter of the pentagon *outside* the circle = $5\times2\tan36$ = $10\tan36$.
This gives $10\sin36<2\pi<10\tan36$ or $5\sin36<\pi<5\tan36$, giving 2.94< π <3.63.

In general, for an n-sided polygon we obtain $n\sin\left(\frac{180}{n}\right)<\pi<n\tan\left(\frac{180}{n}\right)$, so putting n = 100 gives 3.1411< π <3.1426.

Using perimeters of regular polygons gives a narrower approximation for π for a given value of n than using areas.

This method is interesting, but using a calculator to find values of sin *,* cos *and* tan *isn't really valid because the calculator uses its value of* π *to work out them out!*

2.6.9 Rabbit Run. An investigation involving similar work to the above approximating π methods. I want to make a rabbit run area in my garden and I want to make the area as large as possible. (My garden is very big.) I have 24 metres of plastic fencing and I want to know what shape run will enclose the maximum possible area.
Start with rectangular runs.

The rectangular case is an investigation in section 2.2.7.
As shown there, there are 6 rectangles with integer sides and perimeter 24 m.

rectangle	*area*	*rectangle*	*area*
1 × 11	*11*	*2 × 10*	*20*
3 × 9	*27*	*4 × 8*	*32*
5 × 7	*35*	*6 × 6*	*36*

The one which encloses the maximum area is the square one (6 × 6).

What if you set up the run (still rectangular) against the garden fence (then the fencing has to go round only three sides)?

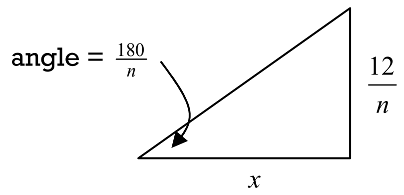
Answer: best option is a 6 × 12 run.
(The function $A = x(24-2x)$ has its maximum value when $x = 6$; could draw a graph.)

What if you use the *corner* of the garden, so that the run (still rectangular) only has to have fencing on *two* of the sides?

Answer: best option this time is a 12 × 12 run.
(The function $A = (12+x)(12-x)$ has its maximum value when $x = 0$.)

What if you are allowed to make any shape run you like. Which will enclose maximum area?
Start in the middle of the garden (away from the garden fence) and find the best shape.
Then try it next to the fence.

For an n-sided regular polygon in the middle of the garden, we can put a point in the centre and join it to each of the vertices to obtain n congruent isosceles triangles. We can divide each of these into two congruent right-angled triangles (below).



We can work out that $x = \dfrac{\frac{12}{n}}{\tan\frac{180}{n}} = \dfrac{12}{n\tan\frac{180}{n}}$*, so the total area of the polygon (*$2n$ *of these right-angled triangles) is* $2n\times\frac{1}{2}x\dfrac{12}{n} = 12x$

$= \dfrac{12\times12}{n\tan\frac{180}{n}} = \dfrac{144}{n\tan\frac{180}{n}}$.

As n gets larger and larger, this area gets closer and closer to the area of a circle with circumference of 24 m.
Radius = $\frac{24}{2\pi} = 3.82$ *m, so area =* $r^2\pi = 45.84$ *m²*.
The ***isoperimetric theorem*** *says that a circle encloses the maximum possible area for a given perimeter. (This is why soap bubbles are spherical.)*

Using the garden fence as just one of the sides, a semicircle would be best. The curved portion would have length 24 m, so the run would be half of a circle of circumference 48 m, so the radius would be $\frac{48}{2\pi}$ *= 7.64 m, and the area would be* $\frac{1}{2}r^2\pi = 91.67$ *m²*.
By using the corner of the garden you could do even better. The run would be a quadrant, the radius would be $\frac{4\times24}{2\pi}$ *= 15.28 m and the area would be* $\frac{1}{4}r^2\pi = 183.35$ *m²*.

2.6.10 **NEED** various equipment. Trigonome-tree! Calculate the height of a tree in the school grounds (or nearby) using a clinometer (a device for measuring angle of elevation – you can make a reasonable one by sticking a protractor onto a piece of cardboard and using a ruler as the rotating portion).

Could estimate the volume of the tree and perhaps estimate how many paper towels (from school toilets) could be made from one tree.

Choose a reasonably tall and thick tree. Assume it is cylindrical (or conical) and measure the circumference with a tape measure.

2.6.11 True or false? Try them with angles.

1. $\sin(x+y)=\sin x+\sin y$
2. $\sin 2x = 2\sin x$
3. $\sin 2x = 2\sin x\cos x$
4. $\sin(90-x)=\sin x$
5. $\sin(180-x)=\sin x$

… and similar with $\cos$ and $\tan$.
Make up some more like these.
Try to prove those that are true by using graphs of the functions.

Answers:
1. False unless x or $y=0$ °, 360°, etc.;
2. False unless $x=0$ °, 360°, etc.;
3. Always true;
4. False unless $x=45$ °, 225°, etc.;
5. Always true.

2.6.12 Trigonometry Facts and Formulas (see sheets).

2.6.13 Methane (CH_4) is a tetrahedral molecule. The carbon atom is surrounded by four hydrogen atoms that get as far away from each other as they can while staying the same distance from the carbon atom.
Calculate the H-C-H bond angle.

Could use molecular models (science dept.).

You could use vectors to solve this problem.

Answer: 109.5° (see sheet)
(This is a very well-known value among chemists, although few could calculate it geometrically!)

You can imagine the hydrogens on the surface of a sphere with its centre at the carbon.
If one or more of the H's are replaced by a different atom, the bond angles won't all be exactly equal any more (unless all four H's are replaced; e.g., in tetrachloromethane, CCl_4).

2.6.14 Find out in which Sherlock Holmes story the detective has to use trigonometry to solve the mystery.

Answer: "The Adventure of The Musgrave Ritual" from "The Memoirs of Sherlock Holmes" by Arthur Conan Doyle.

He really just uses similar triangles. "If a rod of 6 feet threw a shadow of 9 feet, a tree of 64 feet would throw one of 96 feet." This enables him to locate some hidden treasure and solve the case.

2.6.15 Cinema/Theatre Seats.
What is the optimum slope of the floor in a cinema or theatre so that everyone can see over the heads of the people in front?

You could discuss the way that these sorts of calculations using "averages" across the population can fail to satisfy anyone: seats designed for an "average" person may be uncomfortable for almost everyone.
You could estimate how awkward it might be for someone who is very tall or very short. Who would be worse off?

Of course, this won't work if the person in front is wearing a hat!

Estimate the maximum slope that you think would be safe, bearing in mind that the floor will get wet when it's raining outside.
What about someone using a wheelchair?

Answer:

$$\tan^{-1}\frac{\textbf{average eye-to-top-of-head distance}}{\textbf{average thigh-to-knee distance}}$$

If you imagine someone sitting directly behind another person, their eyes need to be above the top of the head of the person in front.
The seat needs to go back at least as far as the distance from their knee to the top of their leg.

Putting in estimates for these gives an angle of about $\tan^{-1}\frac{15}{60}=14$ ° approximately.

This is quite a steep slope to walk up: you would be unlikely to want to go as steep as 20°. It may not be convenient for wheelchair users.

Trigonometry Investigation

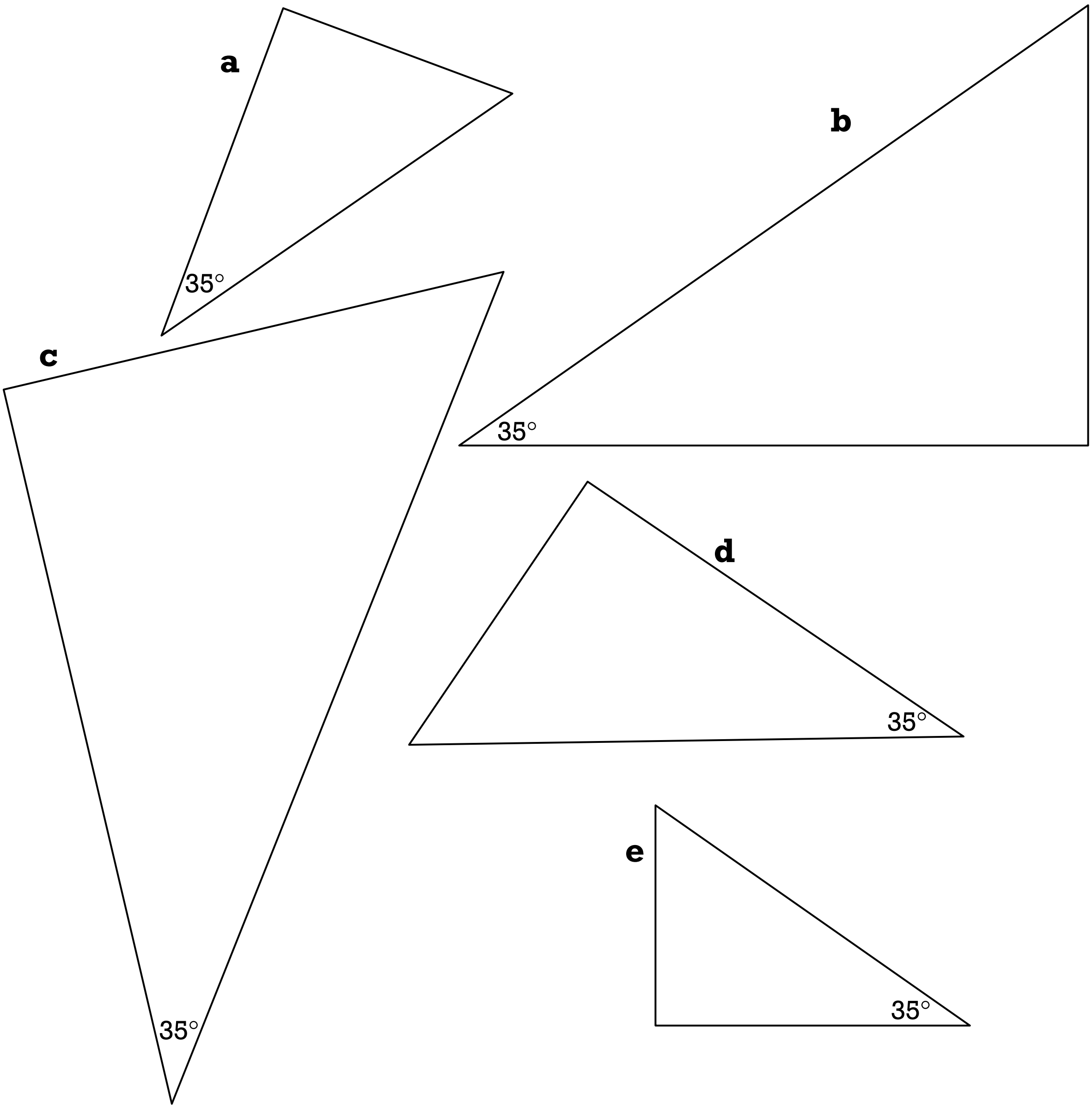
Label the sides of these right-angled triangles with

hyp – hypotenuse (the side opposite the right-angle)

opp – opposite (the side opposite the 35° angle)

adj – adjacent (the side next to the 35° angle)

Then measure the **opp** and **adj** sides with a ruler (to the nearest mm) and complete the table below.
What do you notice?



triangle	**length of side *opposite* to the 35° angle** (mm, to nearest integer)	**length of side *adjacent* to the 35° angle** (mm, to nearest integer)	$\frac{\textbf{opposite side}}{\textbf{adjacent side}}$ (to 2 decimal places)
a			
b			
c			
d			
e			

Trigonometry Facts

LEARN EVERYTHING ON THIS PAGE. LEARN EVERYTHING ON THIS PAGE. LEARN EVERYTHING ON THIS PAGE.

Definitions

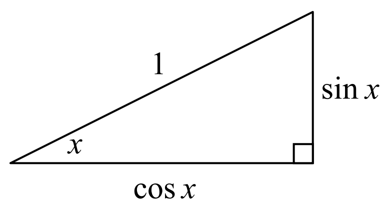
SOHCAHTOA gives the meaning of $\sin x$, $\cos x$ and $\tan x$.

Just remember the other three as

$$\cot x = \frac{1}{\tan x},\ \sec x = \frac{1}{\cos x} \text{ and } \operatorname{cosec} x = \frac{1}{\sin x}.$$

(the ***s*** and the ***c*** go together – ***s***ec with ***c***os and ***c***osec with ***s***in)

Imagine a right-angled triangle with a hypotenuse of 1 and an angle x.
Using SOHCAHTOA, the shorter sides are $\cos x$ and $\sin x$.



So $\tan x \equiv \dfrac{\sin x}{\cos x}$,

and by Pythagoras,

$$\sin^2 x + \cos^2 x \equiv 1.$$

Dividing by $\sin^2 x$ gives $1 + \cot^2 x \equiv \operatorname{cosec}^2 x$, and
dividing by $\cos^2 x$ gives $\tan^2 x + 1 \equiv \sec^2 x$.

Radians

Always use them for angles unless the context specifically uses degrees. Calculus will work only in radians.

360° = 2π, 180° = π, 90° = $\frac{\pi}{2}$, 45° = $\frac{\pi}{4}$, etc.

(the ***3*** and the ***6*** go together in 30° = $\frac{\pi}{6}$ and 60° = $\frac{\pi}{3}$)

Make sure your calculator is in the right mode!

Exact Values

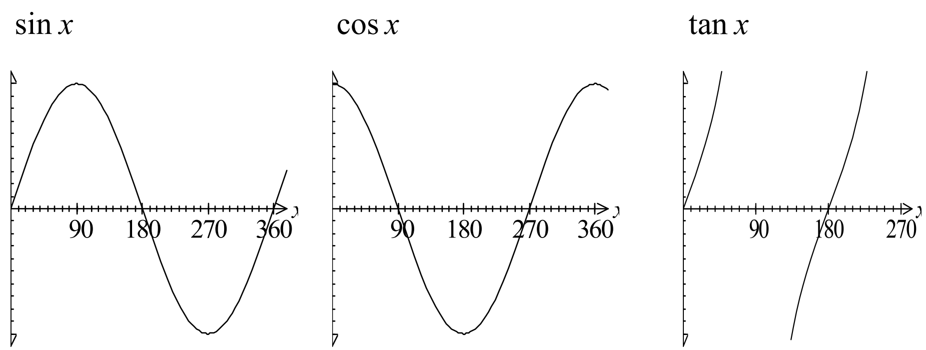
The properties of equilateral and right-angled isosceles triangles give these exact values of $\sin x$, $\cos x$ and $\tan x$.
Use them whenever possible so that your answers are exact.

	0° = 0	30° = $\frac{\pi}{6}$	45° = $\frac{\pi}{4}$	60° = $\frac{\pi}{3}$	90° = $\frac{\pi}{2}$
$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan x$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$\pm\infty$

Notice that $\sin x = \cos(90 - x)$ and $\cos x = \sin(90 - x)$.

Graphs

When you're solving trig equations, finding solutions within a certain range of angles, always draw one of these graphs.



$\sin x$ and $\cos x$ vary between 1 and –1 with a period of 360°.
$\tan x$ can take any value and has a period of 180°.

Use the symmetry of these graphs to find multiple solutions.

Trigonometry Formulas

Compound Angle Formulas

These are the most important ones – you must learn them.

$$\sin(A+B) \equiv \sin A\cos B + \cos A\sin B$$
$$\sin(A-B) \equiv \sin A\cos B - \cos A\sin B$$

"sin,cos *plus* cos,sin"

$$\cos(A+B) \equiv \cos A\cos B - \sin A\sin B$$
$$\cos(A-B) \equiv \cos A\cos B + \sin A\sin B$$

"cos,cos *minus* sin,sin"

$$\tan(A+B) \equiv \frac{\tan A + \tan B}{1 - \tan A\tan B}$$
$$\tan(A-B) \equiv \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

Remember the *minus* in the $(A+B)$ formulas for $\cos$ and $\tan$.

Double Angle Formulas

These come from letting $B = A$ in the above.

$$\sin 2A \equiv 2\sin A\cos A$$
$$\cos 2A \equiv \cos^2 A - \sin^2 A$$
$$\tan 2A \equiv \frac{2\tan A}{1-\tan^2 A}$$

These are also worth learning separately because they're so useful.

Formulas Using Double Angles

These come from combining the double angle formulas with the identity $\sin^2 A + \cos^2 A \equiv 1$.

$$\cos 2A \equiv 2\cos^2 A - 1$$
$$\cos^2 A \equiv \tfrac{1}{2}(1+\cos 2A)$$
$$\cos 2A \equiv 1 - 2\sin^2 A$$
$$\sin^2 A \equiv \tfrac{1}{2}(1-\cos 2A)$$

You could work these out (or look them up) when you need them, but you need to know that they exist.

Factor Formulas

$$\sin A + \sin B \equiv 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$
$$\sin A - \sin B \equiv 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$
$$\cos A + \cos B \equiv 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$
$$\cos A - \cos B \equiv -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

These are very useful, and you certainly don't want to have to work them out.
Either learn them or rely on looking them up when you need them.

Remember the *"minus* $\sin$ *s"* in the last one.

Formulas for Triangles

a, b and c are the lengths of the sides;
A, B and C the angles opposite the 3 sides.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2r$$

This is the *Sine Rule*.
r is the radius of the circumscribed circle.

$$a^2 = b^2 + c^2 - 2bc\cos A$$

This is the *Cosine Rule*.

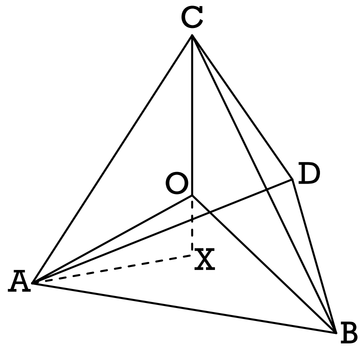
$$\textbf{area} = \tfrac{1}{2}ab\sin C = \sqrt{s(s-a)(s-b)(s-c)}$$

s is the semi-perimeter $= \tfrac{1}{2}(a+b+c)$.

Calculating the H-C-H bond angle in Methane (CH_4)

(Uses Pythagoras' Theorem)

Imagine a tetrahedron ABCD with unit edge length and a hydrogen atom at each of the vertices. (In the diagram below, ABD is the base.)
The position O is the location of the carbon atom, and is equidistant from each of the vertices.
We wish to find angle AOB.



First we can find AX.
Looking from above, X is just the centre of the bottom equilateral triangle, so using trigonometry in this triangle we find that AX = $\frac{1}{2} \div \cos 30 = \frac{1}{\sqrt{3}}$.

Now the total height of the tetrahedron, CX, is a shorter side of the right-angled triangle ACX, so using Pythagoras' Theorem we find that CX = $\sqrt{AC^2 - AX^2} = \sqrt{1 - \left(\frac{1}{\sqrt{3}}\right)^2} = \sqrt{\frac{2}{3}}$.

We next find the C-H bond length l , which is equal to OA or OC.
Using Pythagoras' Theorem in triangle OAX we have

$$\begin{aligned} OA^2 &= AX^2 + OX^2 \\ &= AX^2 + (CX - OC)^2 \\ &= \tfrac{1}{3} + CX^2 - 2 \times CX \times OC + OC^2 \\ l^2 &= \tfrac{1}{3} + \tfrac{2}{3} - 2\sqrt{\tfrac{2}{3}}l + l^2 \end{aligned}$$

So $2\sqrt{\frac{2}{3}}l = 1$ and so $l = \sqrt{\frac{3}{8}}$.
(Subtracting l^2 from both sides is OK here.)

Finally we use the cosine rule in triangle OAB to find angle AOB.

$$\begin{aligned} \cos AOB &= \frac{OA^2 + OB^2 - AB^2}{2 \times OA \times OB} \\ &= \frac{2l^2 - 1^2}{2l^2} = \frac{2\left(\frac{3}{8}\right) - 1}{2\left(\frac{3}{8}\right)} = \frac{-\frac{1}{4}}{\frac{3}{4}} = -\tfrac{1}{3} \end{aligned}$$

So angle AOB = $\cos^{-1}(-\frac{1}{3}) = 109.47$ °.

There are other ways of arriving at the value of this angle.

2.7 Pythagoras' Theorem

- You could say that this is really "Trigonometry" because it's to do with solving triangles.
- It's a good opportunity to revise circles, because there are so many good problems applying Pythagoras' theorem to circles, arcs, spheres, etc. (see later).
- Hypotenuse is the side opposite the right-angle. This may be a better definition than "the longest side", because this way it's clear that there isn't one in a non-right-angled triangle – any scalene triangle and some isosceles triangles will have a "longest" side.)
- Instead of writing $a^2+b^2=c^2$ or $a^2=b^2+c^2$ and having to remember which letter is the hypotenuse, pupils could write it as $\text{hyp}^2=a^2+b^2$.
- The *converse* of Pythagoras is sometimes omitted, but provides a good opportunity to discuss the concept of "converse" and to think of examples of when the converse of something is true and when it isn't. If A and B are the following statements, A = "the triangle is right-angled", B = "the square of the hypotenuse is equal to the sum of the squares on the other two sides", then Pythagoras' theorem is the conclusion that A implies B ($A \Rightarrow B$). The converse is that $B \Rightarrow A$, that any triangle in which statement B is true must be right-angled. So in this case $A \Leftrightarrow B$ (A is equivalent to B), but in general if $A \Rightarrow B$, B doesn't necessarily imply A. One example is if A = "the triangle is right-angled" and B = "the shape has exactly three sides". Here $A \Rightarrow B$ but $B \not\Rightarrow A$ because although all right-angled triangles have three sides, not all triangles are right-angled.
- In three dimensions $a^2=b^2+c^2+d^2$. This makes sense by seeing that b^2+c^2 is the square of the hypotenuse of the right-angled triangle in the plane defined by sides b and c (the plane perpendicular to side d). Then applying 2-d Pythagoras' theorem again gives the result. (You could just as well start with c and d or with b and d .)
- Pythagoras' Theorem is so powerful because it is readily applied to more complicated circumstances than a single right-angled triangle; e.g., any non-right-angled isosceles triangle can be cut into two congruent right-angled triangles.
- In solving right-angled triangles, it's helpful to distinguish between finding the *hypotenuse* (square, add, square root) and finding one of the shorter sides (sometimes called *legs*) (square, subtract, square root).

2.7.1 **NEED** 1 cm × 1 cm squared dotty paper (see section 2.1). Tilted Squares. We are going to draw tilted squares on square dotty paper so that each vertex lies on a dot. Start by drawing a "$\begin{pmatrix}2\\1\end{pmatrix}$ square", where the lowest side goes 2 along and 1 up (gradient of $\frac{1}{2}$). Work out the area.	*Answer:* *If pupils get stuck drawing the squares, they can say "2 along-1 up, 1 along-2 up, 2 along-1 down, 1 along, 2 down" as they go around the square.* *area = 5 cm².*
Try other $\begin{pmatrix}x\\1\end{pmatrix}$ squares and look for a pattern.	*There are various ways of cutting up the square. (Some pupils may prefer to measure the sides with a ruler as accurately as they can and find the area that way, although it may be less "elegant".)* *area =* x^2+1
Then try $\begin{pmatrix}x\\y\end{pmatrix}$ squares.	*area =* x^2+y^2
Then focus in on the right-angled triangle "underneath" the square. What are these results telling us about the sides of the triangle?	*If the area of a square is 36 cm², then how long are the sides? etc.*

2.7.2 See if Pythagoras' Theorem works for all triangles (see "Triangles and Tilted Squares" sheet).

Answers:

A	obtuse-angled (25 > 9 + 10)
B	right-angled (26 = 8 +18)
C	acute-angled (16 < 13 + 13)
D	right-angled (25 = 5 + 20)
E	obtuse-angled (9 > 2 + 5)

The square on the longest side equals the sum of the squares on the other two sides only if the triangle is right-angled*.*
If it is obtuse-angled, then the square on the longest side (the one opposite the obtuse angle) is greater than the sum of the squares on the two other sides.
If it is an acute-angled triangle, then the square of any of the sides is smaller than the sum of the squares on the other two.

Try other shapes on the sides of right-angled triangles; e.g., semicircles or equilateral triangles.

Any shape will work provided that the shapes on each side are mathematically similar. This corresponds to multiplying the equation $\text{hyp}^2 = a^2 + b^2$ by a constant.

2.7.3 How many different sized squares can you draw on a 3 × 3 dotty grid if every vertex has to lie on a dot?
What if you use a larger square grid?

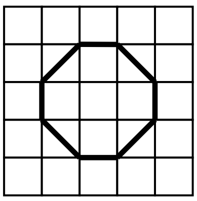
(See the similar investigation using triangles and quadrilaterals in "Polygons" section.)

Answer: 3
For an $n \times n$ *grid of dots,*

n	***no. of different sized squares***
1	*0*
2	*1*
3	*3*
4	*5*
5	*8*
6	*11*
7	*15*
8	*18 (yes)*

The pattern is broken at $n = 8$ *because the tilted squares with sides of gradient* $\pm\frac{3}{4}$ *and* $\mp\frac{4}{3}$ *have sides with length 5 units, matching one of the untilted squares already counted. This will happen whenever you get Pythagorean Triples.*

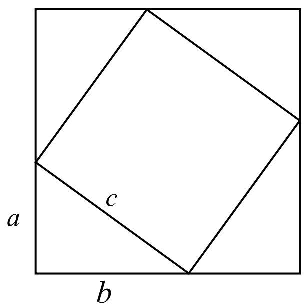
2.7.4 Why is this shape *not* a regular octagon?



Answer:
Although all the interior angles are all 135°, there are four sides of unit length and four "diagonal" sides of longer length $\sqrt{2}$ *units.*

This question aims to confront a common misconception.

2.7.5 Proof: there are a vast number of them.
The simplest is probably the one equating areas in the diagram below (see right).



This is a square with sides of length c inside a square with sides of length $a+b$.
The area of the large square can be worked out in two different ways, and they must give the same answer.

First method: area = $(a+b)^2 = a^2 + 2ab + b^2$.
(If pupils are not familiar with this result from algebra, you can divide up the large square into two congruent rectangles, each of area ab, and two squares of different sizes, a^2 and b^2.)

Second method: add up the areas of the smaller square and the four congruent right-angled triangles; so
area = $c^2 + 4 \times \frac{1}{2}ab = c^2 + 2ab$.
So $a^2 + b^2 + 2ab = c^2 + 2ab$, and therefore $a^2 + b^2 = c^2$ (Pythagoras' Theorem).

There are a large number of different proofs of Pythagoras' Theorem: pupils could search on the internet for other proofs.

2.7.6 Pythagorean Triples.
The lengths of the sides of integer right-angled triangles.
(3,4,5); (5,12,13) and (7,24,25) are the simplest.
Any numbers ($2pq$, p^2-q^2, p^2+q^2) will always work (where p and q are integers and $p>q$).

A method equivalent to this for generating Pythagorean triples is to add the reciprocals of any 2 consecutive odd or consecutive even numbers. The numerator and denominator of the answer (whether simplified or not) are the two shorter sides of a right-angled triangle; e.g.,
$\frac{1}{6}+\frac{1}{8}=\frac{7}{24}$, giving the (7,24,25) triangle.

Some pupils could write a BASIC (or similar) computer program to find Pythagorean Triples.

A "primitive" Pythagorean Triple is one that isn't just a scaling up of a similar smaller triangle by multiplying all the sides by the same amount; i.e., it's one in which the sides are pairwise co-prime. To get only these, p and q must be co-prime and of opposite "parity" (one odd, the other even).

You can prove that these sides satisfy Pythagoras' formula by squaring and adding:
$(p^2-q^2)^2+(2pq)^2$
$=p^4-2p^2q^2+q^4+4p^2q^2$
$=(p^2+q^2)^2$

This works because $\frac{1}{p-1}+\frac{1}{p+1}=\frac{p+1+p-1}{p^2-1}=\frac{2p}{p^2-1}$, as above where $q=1$.

2.7.7 Pythagorean Triples.
1. Investigate the factors of the numbers in "primitive" Pythagorean triples.
2. Investigate the product of *the two legs* of a Pythagorean triple.
   Try for different Pythagorean triples.
3. Investigate the product of *all three sides* of a Pythagorean triple.
   Try for different Pythagorean triples.

Answers:
1. *In "primitive" triples (see above), the largest number is always odd, and of the other two one is odd and one is even. One of the sides is always divisible by 2, one by 3 and one by 5 (see below). (These may all be the same side; e.g., 60 in 11, 60, 61.)*

2. *The product of the two legs is always a multiple of 12 (or, equivalently, the area is always a multiple of 6). (Interestingly, this area can never be a square number.)*
   You can prove this using the expressions above:
   area = $\frac{1}{2}\times 2pq\times(p^2-q^2)=pq(p^2-q^2)$.
   Here, either p is even or q must be, so pq is a multiple of 2. If p or q are multiples of 3, then the whole thing will be a multiple of 6. If neither p nor q is a multiple of 3, then $p=3r\pm1$ and $q=3s\pm1$, so p^2-q^2 must be a multiple of 3 (multiply out p^2 and q^2 and the "+1's" cancel out), so the whole area is still a multiple of 6. Because the product of the legs is $2pq(p^2-q^2)$, then this number will be a multiple of 12.

Answers (continued):
3. *The product of all three sides is always a multiple of 60. To show this, in addition to the above argument we need to show that $pq(p^2-q^2)(p^2+q^2)$ is a multiple of 5.*
   If either p or q is a multiple of 5, then clearly the whole thing will be. If neither is, then $p=5r\pm1$ or $p=5r\pm2$ and $q=5s\pm1$ or $q=5s\pm2$. When you multiply out p^2 and q^2, you find that they are either 1 more or 1 less or 4 more or 4 less than multiples of 5. So it is always the case that either p^2+q^2 or p^2-q^2 is a multiple of 5, so the whole thing must be.

2.7.8 Fermat's Last Theorem.
We have some solutions to $a^2+b^2=c^2$ where a, b and c are positive integers (see Pythaogrean Triples above).
Try to find solutions to these equations:
$a^3+b^3=c^3$
$a^4+b^4=c^4$

Pupils could look for solutions to $a^2+b^2+c^2=d^2$.

Fermat (1601-1665) believed that there were no solutions to the equation $a^n+b^n=c^n$ where $n>2$ and a, b and c are positive integers. He claimed to have a proof but never wrote it down. It has since been proved using highly complicated maths.

There are many "Pythagorean Quadruples"; e.g., (1, 2, 2, 3); (1, 4, 8, 9); (9, 8, 12, 17). This time, any numbers ($2pr$, $2qr$, $p^2+q^2-r^2$, $p^2+q^2+r^2$) work, where $p,q,r>0$.

2.7.9 **NEED** keyboard diagrams (see sheets). Keyboard Typing.
Imagine typing words with 1 finger. Say that each key is 1 cm × 1 cm. How far does your finger have to move to type certain words? (Calculate from the centre of each key.)
e.g., a word like FRED is easy.
What about HELP?
What four-letter word has the longest distance on the keyboard?

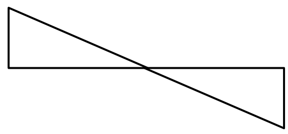
A different way of thinking of "word length". (The keys on the diagram are 1.5 cm × 1.5 cm to discourage measuring.)

Answers:
FRED = 3 cm (F-R, R-E and E-D)

HELP = 3.64 + 6.58 + 1.12 = 11.34 cm (using Pythagoras' Theorem)
ZONE = 14.58 cm, but there may be longer words.

2.7.10 Two football players start side by side. They each run 4 m in a straight line, turn 90° to the right and run another 3 m, again in a straight line. What is the furthest apart they could now be?

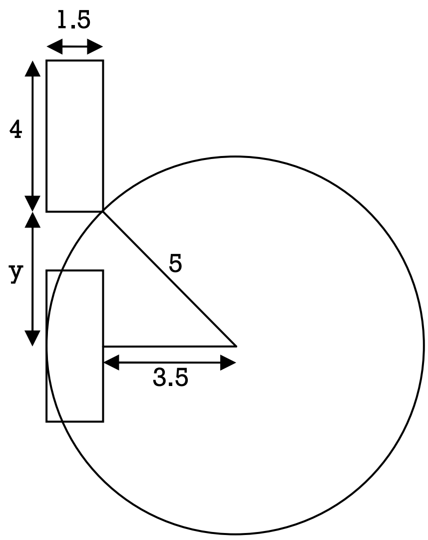
Answer: 10 m, if they started facing in opposite directions. (Imagine two 3-4-5 right-angled triangles meeting at the players' starting point.)



2.7.11 A straight road contains a row of parked cars all of width 1.5 m and length 4 m. If one of the cars has a turning circle of 10 m, how much space will it need in front of it so that it can pull out without having to reverse?

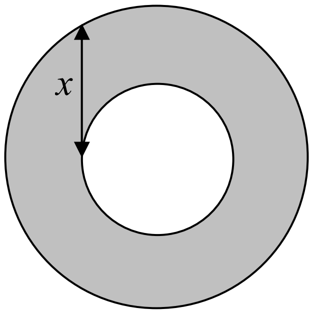
What assumptions do you have to make?
(A turning circle of 10 m kerb-to-kerb means that the car can just manage a U-turn at slow speed in a street 10 m wide.)

Answer:
In the diagram to the right, the distance from the centre of the turning circle to the offside of the car is 5 – 1.5 = 3.5 m. Applying Pythagoras' Theorem, $y^2 = 5^2 - 3.5^2$*, giving* y *= 3.6 m, measured from the mid-point of the length of the car. The necessary distance in front of the car is therefore 3.6 – 2 = 1.6 m.*



We have assumed that the cars are parked exactly in line, that both cars have the same width and that the driver gets full right-lock as soon as the car begins to move.

2.7.12 In the diagram below, find the shaded area in terms of x.



Answer:
If the large circle has radius R *and the small circle,* r*, then the shaded area =* $\pi R^2 - \pi r^2 = \pi(R^2 - r^2)$.
But by Pythagoras' Theorem, $x^2 = R^2 - r^2$*, so the shaded area =* πx^2.

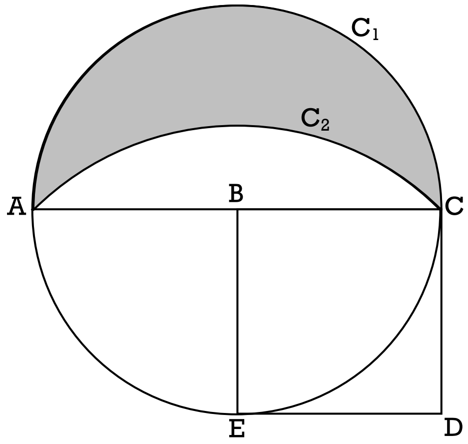
2.7.13 A ladder of length 13 feet is standing upright against a wall. If the top end slides down the wall 1 foot, how far out from the wall will the bottom end move?

Answer:
5 feet (a 5-12-13 right-angled triangle)

2.7.14 A cable 1 km long is lying flat along the ground with its ends fixed. If its length is increased by 1 m but the ends are still fixed 1 km apart, how high up can the mid-point of the cable be raised before it becomes taut? (Assume the cable doesn't stretch or sag.)

Answer: 22.4 m (a surprisingly large amount). The shape produced is a (very) obtuse-angled isosceles triangle. Each half is a right-angled triangle with hypotenuse 500.5 m and base 500 m. Calculating the third (vertical) side gives the answer.

2.7.15 In the diagram below, C_1 is a semicircular arc centred on B and C_2 is a quarter-circular arc centred on E. Prove that the area of the shaded *lune* between C_1 and C_2 is equal to the area of the square BCDE.



Answer:
Let BC = r.
Then the area of the square BCDE = r^2.
Area of semicircle $C_1 = \frac{1}{2}\pi r^2$, *and area of quadrant ACE =* $\frac{1}{4}\pi(\sqrt{2}r)^2 = \frac{1}{2}\pi r^2$.
Area of triangle ACE = $\frac{1}{2}(\sqrt{2}r)^2 = r^2$ *(since angle AEC = 90°, angle in a semicircle), so area of segment =* $\frac{1}{2}\pi r^2 - r^2$.
Therefore area of shaded lume =
$\frac{1}{2}\pi r^2 - (\frac{1}{2}\pi r^2 - r^2) = r^2$ *= area of square BCDE, as required.*

This and other similar results were discovered by Hippocrates of Chios (470-410 BC).

2.7.16 A rope is attached to the top of a vertical pole and at the bottom 1 m is lying on the ground. When the end of the rope is pulled along the ground until it is taut, its end is 5 m from the base of the pole. How long is the rope and how high is the pole? (The rope doesn't stretch.)

Answer: The rope is 13 m long and the pole is 12 m high (5, 12, 13 triangle).
If h *is the height of the pole, then* $(h+1)^2 = h^2 + 5^2$, *and solving this equation gives these values.*

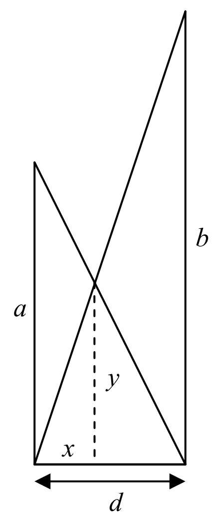
2.7.17 A narrow passageway of width 1 m contains two ladders leaning against the walls. Each has its foot at the bottom of one wall and its top at the top of the other wall. If the walls have heights 2 m and 3m, how high above the ground is the point where the ladders cross?

Answer: 1.2 m
One approach is to model the ladders as segments of the lines $y = 3x$ *and* $y = -2x + 2$.
Solving simultaneously gives $x = 0.4, y = 1.2$.

Alternatively, you can use similar triangles. Using the letters as defined on the right,

$\frac{y}{x} = \frac{b}{d}$ *and* $\frac{y}{d-x} = \frac{a}{d}$, *so eliminating y,*

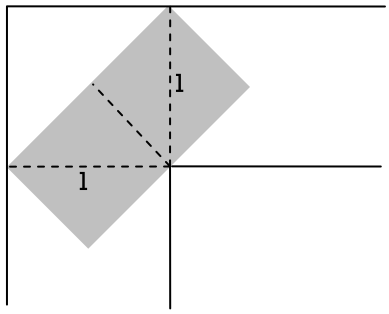
$\frac{bx}{d} = \frac{a(d-x)}{d}$ *so that* $x = \frac{ad}{a+b}$ *and* $y = \frac{ab}{a+b}$, *so the answer to the original question didn't depend on* d *(1 m).*



Similar-looking problems where the lengths of the ladders are given instead of vertical heights are much harder, often leading to quartic equations.

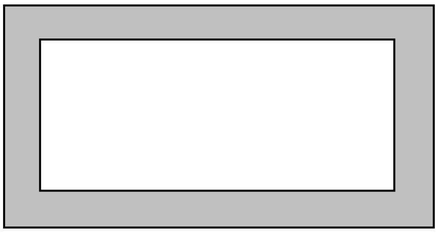
2.7.18 A piano in a cuboid crate has to be moved round a right-angled corner in a corridor of width 2 m. If no part of the crate is lifted off the floor, what are the dimensions of the biggest crate that will just go round the corner? Assume that the floor is perfectly horizontal and the walls perfectly vertical.

Answer: The tightest squeeze will happen when the crate is positioned as below.

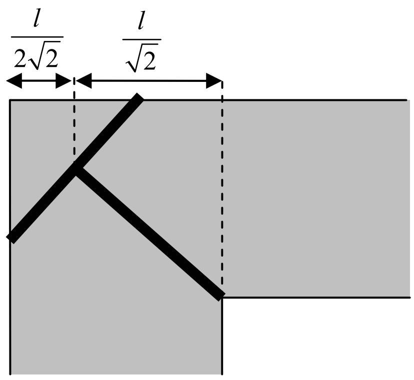


If *the crate just fits, then its sides will be* $\frac{\sqrt{2}}{2}$ *by* $\sqrt{2}$ *, so it's area will be 1 m²*.

2.7.19 A border of 1 m width around a rectangular garden is covered with wet cement. You have two 98 cm long narrow planks of wood. Can you use them to bridge across the cement from the outside to the inside?



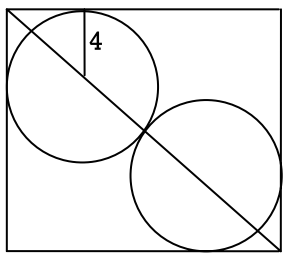
Answer:
Just, if you put them across at the corner. If l *is the length of the planks, and they just meet in the arrangement shown below, then the distances marked are as shown.*



Therefore, $\frac{l}{2\sqrt{2}}+\frac{l}{\sqrt{2}}=1$*, giving* $\frac{3l}{2\sqrt{2}}=1$*, so*
$l=\frac{2\sqrt{2}}{3}=0.94$ *m (2 dp).*
But you would need to allow a bit for overlapping of the planks and the grass.

2.7.20 Four identical apples of diameter 8 cm have to be fitted into a cubical box. What is the smallest box that will do? Assume that the apples are perfect spheres.

Answer:
The apples need to be stacked "tetrahedrally" so that there are 2 along the diagonal of the bottom of the box and 2 along the other diagonal in the top half of the box.

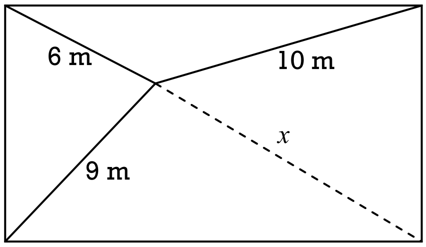


You can see that the diagonal of the box has length $4\sqrt{2}+4+4+4\sqrt{2}$ *, so the sides of the box have length* $\frac{8}{\sqrt{2}}(1+\sqrt{2})=4\sqrt{2}+8$ *= 13.7 cm.*

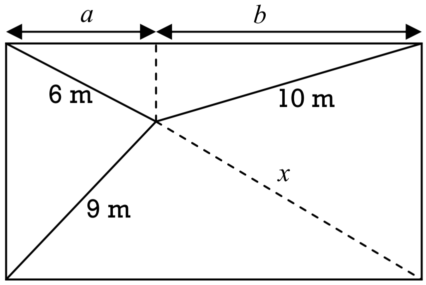
2.7.21 Which fits better, a square peg in a round hole or a round peg in a square hole?

Answer:
A round peg in square hole occupies
$\frac{\pi r^2}{(2r)^2}=\frac{\pi}{4}=79\%$ *of the square, and this is better than a square peg in round hole, which occupies only* $\frac{\left(\frac{2r}{\sqrt{2}}\right)^2}{\pi r^2}=\frac{2}{\pi}=64\%$ *of the circle.*

2.7.22 I am standing in a rectangular hall and my distances from three of the corners are as shown below. How far must I be from the fourth corner?



Answer:
Let a and b be as shown below.



Then using Pythagoras' theorem in the top two right-angled triangles, we can equate expressions for the vertical dashed line, giving $6^2 - a^2 = 10^2 - b^2$.
Similarly for the bottom part of the diagram, $9^2 - a^2 = x^2 - b^2$*, and subtracting these equations leaves* $9^2 - 6^2 = x^2 - 10^2$*, and solving this gives* $x = \sqrt{145} = 12.04$ *m.*

2.7.23 How big is the smallest circle which you can fit a 2 cm by 4 cm rectangle into?

Answer:
The widest length in the rectangle will be the diagonal, which is $\sqrt{2^2 + 4^2} = \sqrt{20}$ *cm, so that will have to be the diameter. So the radius of the circle will be* $\frac{1}{2}\sqrt{20} = \sqrt{5} = 2.24$ *cm.*

Triangles and Tilted Squares

Plot these triangles on axes going from 0 to 20 both horizontally and vertically.
(With a bit of overlapping, they will all fit onto one set of axes.)

Label them A to E, and note what *kind* of triangle each one is.

Draw tilted squares on each side and work out their areas.
Look at your results for each triangle.
What do you notice?

A	(10, 7)	(9, 4)	(6, 4)
B	(9, 15)	(4, 14)	(7, 17)
C	(14, 7)	(17, 5)	(14, 3)
D	(14, 16)	(18, 14)	(17, 12)
E	(2, 3)	(3, 5)	(3, 2)

Q	W	E	R	T	Y	U	I	O	P
A	S	D	F	G	H	J	K	L	
Z	X	C	V	B	N	M	,		

Q	W	E	R	T	Y	U	I	O	P
A	S	D	F	G	H	J	K	L	
Z	X	C	V	B	N	M	,		

Q	W	E	R	T	Y	U	I	O	P
A	S	D	F	G	H	J	K	L	
Z	X	C	V	B	N	M	,		

Q	W	E	R	T	Y	U	I	O	P
A	S	D	F	G	H	J	K	L	
Z	X	C	V	B	N	M	,		

2.8 Loci and Constructions

- Strictly speaking, "constructions" can be done with pencil, compasses and straight-edge only. No measuring with ruler or protractor is allowed. But often this topic gets merged with scale drawing, so this distinction is lost.
- In drawing work it's worth aiming for an accuracy of $\pm 1°$ and $\pm$ 1 mm. A sharp pencil helps.
- When using compasses, it can be useful to have a screwdriver handy for tightening them up.
- For some varied loci to draw, see the sheet.

2.8.1 **NEED** various props (e.g., clock, paper plate, door with handle, large book that will stand up on its end). (See sheet.)
Explain that locus means "all the possible positions that fit a particular rule".
On scrap paper, pupils draw the loci for various situations described by the teacher.

Practical Lesson to introduce the concept.

You can consider that a point moves according to a rule, or if you prefer "points" to be "fixed" then the locus is the set of all points that satisfy a particular condition. It may be that one or other of these perspectives may be more helpful depending on the context of the particular locus problem.

2.8.2 **NEED** outdoor or large indoor space (playground, hall, gym, etc.) People maths.
Follow these instructions, then try to describe the loci.
It's important to be able to describe precisely the lines that you get.

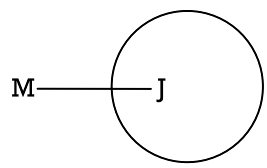
1. Leo, stay where you are. Everyone else, get 2 m away from Leo. ("Are you 2 m away from Leo? Could I stand here?")
2. Max and Jo, stay where you are. Everyone else, stand somewhere where you are the same distance from Max as you are from Jo.
3. (Harder) Stand exactly twice as far from Max as you are from Jo. (Need to realise that there are points "in between" M and J and points beyond J but none beyond M.)
4. Get exactly 1 m away from that wall (a long wall).
5. Get exactly 1 m away from that wall/hedge (a shorter wall with ends).
6. Stand so that you're the same distance from Jamie as you are from that wall.

There may be other possibilities according to the space and objects you have available. If you don't have a suitable wall/hedge, you can use chalk to mark a line on the ground.

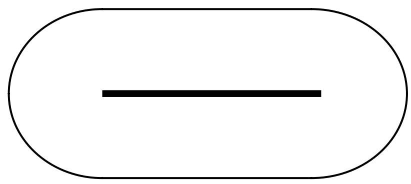
(If it isn't practicable to do this with people, you could move coloured counters or cubes on a desk so as to satisfy these conditions.)

Should produce …

1. *a circle, radius 2 m, centred on Leo;*
2. *the perpendicular bisector of the line segment joining Max and Jo;*
3. *If stuck, the teacher can stand $\frac{2}{3}$ of the way from M to J and show that this point is acceptable or the same distance M is from J the other side of J (along MJ extended). Answer: a circle of radius $\frac{2}{3}$ of the distance between M and J, as shown below;*



4. *a line parallel to the wall, 1 m from it;*
5. *parallel lines along the sides and semicircles round the ends (see below);*



6. *a parabola.*

2.8.3 Think of some examples of loci in everyday life and describe them in words.

e.g., grass watered by garden sprinklers, school catchment areas, bomb blast radii, TV transmitter areas.

2.8.4 **NEED** compasses. The Goat and the Shed.
Using a scale of 1 cm to 1 m, do a scale drawing in the middle of the page of a rectangular shed that is 3 m by 5 m.
Do a plan view (from above).
The shed is surrounded by grass and a goat is tied up to one outside corner of the shed.

1. If the rope is 2 m long, shade in the grass that the goat can eat.
   Calculate the area of grass it can eat.

2. The goat would like more freedom (and grass!), so the rope is replaced by one that is 4 m long.
   On a new drawing of the shed, again shade in the grass that the goat can reach and calculate its area.

3. What happens if the length of the rope is increased to 6 m?

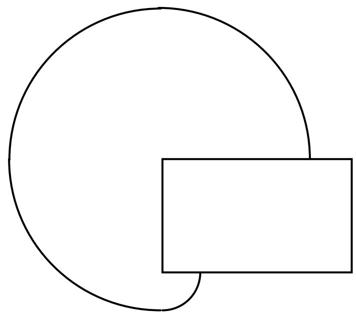
4. How long does the rope have to be to reach all the way round the shed? How much grass can the goat eat then?

5. (Much harder) Finally a 10 m rope is tried. How much grass can the goat eat now?

What assumptions have we made in solving these problems?
We've i*gnored the difference between the place where the rope ends and the "reach" of the goat's mouth; assumed the rope is always horizontal and the walls of the shed are vertical; assumed the rope doesn't stretch or break or get chewed through!; assumed no limits to the area beyond the shed; etc.*

Answers:
1. *Three quarters of a circle of grass around the shed.*
   Area = $\frac{3}{4}\pi 2^2 = 9.42$ *m²*.

2. *Along the 5 m side the situation is similar to before, but along the 3 m side the rope will reach past the next corner of the shed and will catch and the goat can then make an additional quadrant of radius 1 m (see below).*

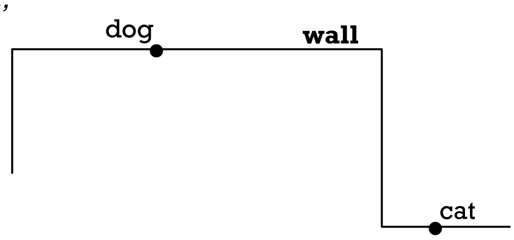


Area = $\frac{3}{4}\pi 4^2 + \frac{1}{4}\pi 1^2 = 38.48$ *m²*.
3. *The rope catches at both corners now, so area =* $\frac{3}{4}\pi 6^2 + \frac{1}{4}\pi 1^2 + \frac{1}{4}\pi 3^2 = 92.68$ *m²*.
4. *8 m to reach right round the shed, and now area =* $\frac{3}{4}\pi 8^2 + \frac{1}{4}\pi 3^2 + \frac{1}{4}\pi 5^2 = 177.50$ *m²*.
5. *Now the same bit of grass can be reached by the goat going clockwise or anticlockwise around the shed, so this method of calculation will overestimate the area.*
   The true answer is about 280 m².

2.8.5 Cats and Dogs.
A cat and a dog don't like each other. They are kept on leads fastened to a wall at separate points. If the dog's lead is 2 m long, draw diagrams to find out where the cat's lead could be fastened and how long it could be so that the two animals would never be able to reach each other.
Try different shaped areas.
Use a scale of 1 cm to 1 m.

Some of the principles of mathematical modelling again.

e.g.,



2.8.6 Ellipse.
Draw a circle with radius 5 cm and mark a point inside the circle about 4 cm from the centre. Fold the circle so that a point on the circumference just touches the marked point. Unfold the paper and draw a line along the fold mark. Repeat for a different point on the circumference. (Keep the point inside the circle the same.)

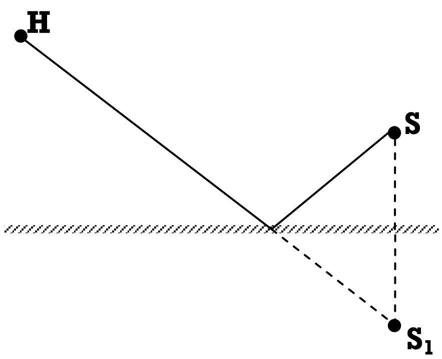
Another method for generating an ellipse is to tie string slack between two fixed points and push the string taut with a pencil. Then draw with the pencil keeping the string tight. The two fixed points are the foci *of the ellipse.*
(See "Loci" sheet.)

Eventually you will end up with an ellipse.

2.8.7 John wants to walk from his house H to school S *via* the river (to feed the ducks) along as *short* a total distance as possible.
What path should he take? (He can feed the ducks just as well at any point along the river.)

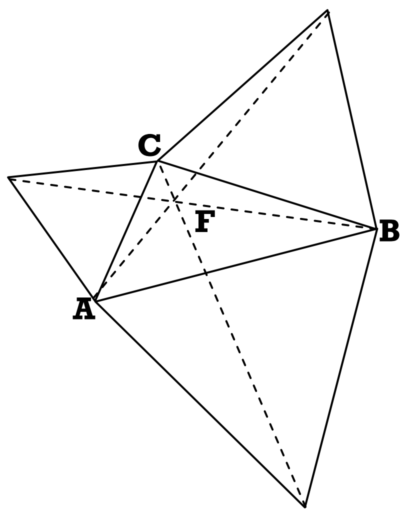
You can fix H and S at certain perpendicular distances from the river, and a certain distance apart, and try by trial and improvement to get the shortest total distance. (e.g., scale 1 cm = 100 m)

Answer:



Light travels via *the shortest route between two points (this is what we mean by "straight"), so imagine that the river is a mirror and H is a light source. To get the reflected beam to go through S, construct* S_1*, the reflection of S in the mirror and join H and* S_1 *with a straight line. Where this intersects the mirror is the point on the river that John should walk to.*

2.8.8 There are three towns A, B and C at different distances from each other. Where should you build roads to connect the three towns so that the minimum length of road is constructed? Do a clear drawing (A, B and C can be the vertices of any scalene triangle) and mark on the exact positions of the roads.



Answer:

Two possibilities:

1. *If one of the angles in the triangle is* $\geq 120°$*, then road should be built from this vertex to the other two;*
2. *If none of the angles is as big as* $120°$*, then you need to find the "Fermat Point", which when joined to each vertex makes three lines all at* $120°$ *to each other. One way of finding this point is to construct equilateral triangles on each side of the original triangle and join the outermost vertices of these to the opposite vertices of the original triangle. Where the lines cross is the Fermat Point. See diagram on the left.*

The Fermat point is F.
The roads should be built along AF, BF and CF.

2.8.9 **NEED** A4 plain paper and sticky tape.
Do an accurate scale drawing of a football pitch using this information.

1. The pitch should be 100 yards long and 50 yards wide.
2. The centre circle should have a radius of 10 yards.
3. The goal is 8 yards wide and surrounded by the 6-yard box (a rectangle 20 yards by 6 yards).
4. The penalty area is the 18-yard box (a rectangle 44 yards by 18 yards).

The penalty point should be 12 yards from the goal line and half way across the width of the pitch.

You could use three sheets of A4 paper taped together and a scale of 1 cm for every 2 yards.

Green paper adds a bit of realism!

Pupils could of course research the measurements for other kinds of pitches and draw those if they prefer.

2.8.10 Drawing accurate triangles.
Use compasses to draw these triangles as accurately as possible.

1. AB = 10 cm; AC = 8 cm; BC = 8 cm;
2. AB = 10 cm; AC = 8 cm; BC = 6 cm;
3. AB = 10 cm; AC = 4 cm; BC = 8 cm;
4. AB = 10 cm; AC = 4 cm; BC = 5 cm.

Pupils can check the accuracy of their drawings by measuring the angles.
Exact values are given on the right.

Answers:

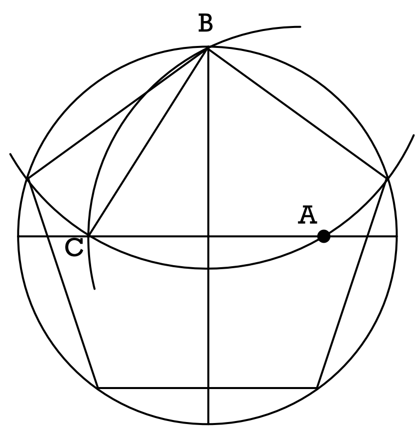
1. *isosceles: A = B = 51.3°; C = 77.4°;*
2. *r-angled: A = 36.9°; B = 53.1°; C = 90.0°;*
3. *obt-ang: A = 49.5°; B = 22.3°; C = 108.2°;*
4. *no such triangle because in any triangle the sum of the two shorter sides must be more than the longest side (the shortest distance from A to B must be the line AB, so AC and BC together must come to more than this, otherwise it won't join up).*

2.8.11 Drawing Polygons with Compasses.
Pupils first need to train their compasses to behave properly, so it's worth starting by making sure everyone can draw a circle and get a single clean smooth line (no wobbles).

This can make good display work.

1. Regular Hexagon – the easiest.
Draw a circle of radius 5 cm in the middle of the page. Keep the compasses open at 5 cm. Put a mark on the circumference at the top of the circle. Put the compass point here and mark off the two points the pencil reaches to on the circle. Repeat at those points until you have 6 equally spaced points round the circle. Join them up with a ruler.

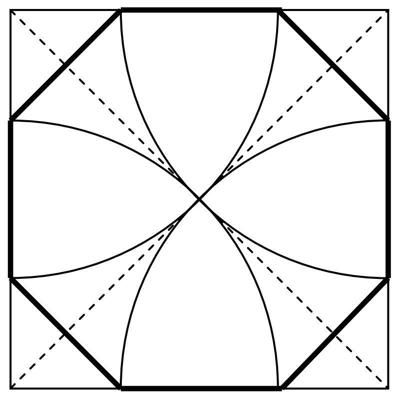
2. Regular Pentagon – need to be good at following instructions! (See diagram right.)
Draw a circle of radius 5 cm in the middle of the page. Draw two perpendicular diameters (horizontal and vertical).
Set the compasses to a radius of 2.5 cm. With centre A (mid-point of horizontal radius, see right) and radius AB draw an arc that cuts the circumference at B and the horizontal diameter at C.
Then draw an arc centred at B with radius BC to cut the circumference twice. This distance (BC) is the length of the side of the regular pentagon, so step this round the circumference and join up the points.



To draw the perpendicular diameters, pupils could use the grid lines if using squared paper; otherwise need to do a perpendicular bisector.

Could bisect the radius to get the 2.5 cm, or "cheat" by measuring with a ruler!

3. Regular Octagon – not too hard.
Draw a square with sides 8 cm and draw in the diagonals. Set the compasses to a radius of half the diagonal (the distance from a vertex to the centre of the square) and draw an arc centred on one of the vertices so that it crosses the two adjacent sides. Repeat for the other three vertices and join up the 8 crossing points with a ruler (see right).



2.8.12 Constructions with compasses:

1. Perpendicular Bisector of a line segment: all the points that are the same distance from one end of the given line as they are from the other;
2. Angle Bisector of two non-parallel lines: the line that makes the same angle with each of the starting lines.

This makes use of the fact that the diagonals of a rhombus bisect each other.
It doesn't matter how far the compasses are opened so long as it's more than half the distance between the two starting points and it doesn't alter.

Loci

Try to sketch these loci as carefully as you can.

1 The locus of a point on the front tip of an aeroplane as it comes in to land.

2 The locus of a white dot of paint on the moving end of the minute hand on a clock. What if the dot of paint is only half way along the minute hand?

3 The locus, viewed from above, of a point on the door handle when I open the door.

4 The locus of a ball thrown over a wall.

5 The locus of a point on the rim of a bicycle wheel as the bicycle moves without slipping along a flat horizontal road at a steady speed.
What would happen if the bicycle speeded up?

6 The locus of a point on the flange of a train wheel as the train moves along a flat horizontal track at a steady speed.

7 The locus of a point part-way along one of the spokes of a bicycle wheel as the bicycle moves along a flat horizontal road at a steady speed.

8 The locus of a point on the circumference of a circle as it rolls (without slipping) around another circle that is the same size.
e.g., try rolling a 10 p coin around another one.

9 The locus of a point whose total distance from two fixed points is a constant.
e.g., tie a slack piece of string between the two points and use a pencil to make the string taut – draw with the pencil, keeping the string taut.

10 The locus of the points all the way along a uniform (same all the way along) rope suspended between two horizontal points high enough so the rope doesn't touch the ground.

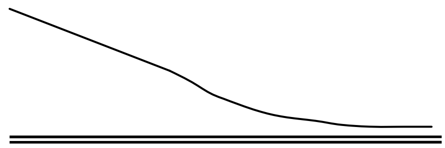
11 The locus of a point on the top right corner of a book as the book "rolls" (without slipping) along a table.
Draw this one as accurately as you can. You could take the book as being 15 cm wide by 20 cm high (i.e., a 3:4 ratio of width to height). Its thickness doesn't matter so long as it is thick enough to stand up without falling over.

12 The locus of a point mid-way along a ladder as the ladder slides down from a vertical position against a wall until it is horizontal. (The top of the ladder slides down the wall; the bottom of the ladder slides along the ground.)

What other loci can you think of and sketch?

Loci ANSWERS

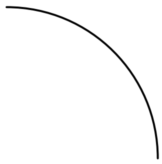
1 Aeroplane landing: something like this



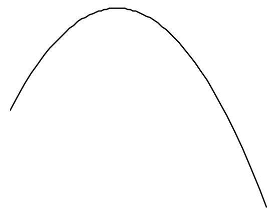
2 Dot on the end of the minute hand of a clock: a circle of radius equal to the length of the minute hand and centred on the middle of the clock.

If the dot were only half way along, the circle would have the same centre but half the radius.

3 Door handle as the door opens: roughly a quarter of a circle. (Some pupils will think it's a straight line.)



4 Ball thrown over a wall:



The curve is part of a *parabola*
The equation is $y = ax^2 + b$, $a > 0$.

5 Bicycle wheel: point on the rim

(You can demonstrate this with a plate marked with a dot and rolled along a table – when the dot touches the table it doesn't slide backwards, so there are no loops.)

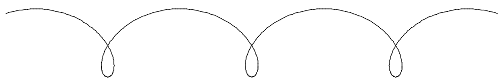
Called a **cycloid.** Looks a bit like a row of semicircles but isn't.
If the bicycle speeded up, it would make no difference to the curve, although the later parts would get drawn faster.

The cycloid has lots of interesting properties; e.g.,

- it's the strongest shape for the arch of a bridge;
- the area under each "hump" is 3 times the area of the bicycle wheel;
- if you turn the shape upside down and roll a marble down the inside it takes the same amount of time to reach the bottom wherever you start it from – it's also the solution to the "brachistochrone problem": what path should a particle roll down to get from one point to a lower point in the shortest possible time?

The parametric equations are $x = r(\theta - \sin\theta)$ and $y = r(1 - \cos\theta)$, where $r > 0$.

6 Train wheel: point on the flange



Called a **prolate cycloid.**

This is the solution to the following puzzle: If a train is travelling from London to Edinburgh, what points on the train are (at a given instant) moving towards London?
(Assume that the train keeps going throughout the journey.)

Answer: points on the flanges of the wheels.
(Passengers walking down the train will still be heading towards Edinburgh because their speed relative to the carriage will be tiny compared with the speed of the train.)

The parametric equations are $x = r\theta - d\sin\theta$ and $y = r - d\cos\theta$, where $d > r$.

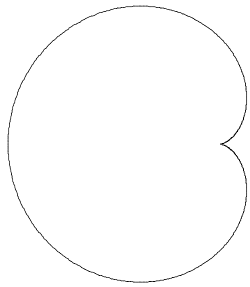
Loci (continued) ANSWERS

7 Bicycle wheel: point part-way along one of the spokes

Called a **curtate cycloid**.

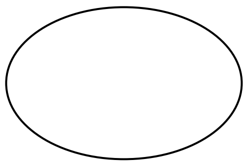
The parametric equations are $x = r\theta - d\sin\theta$ and $y = r - d\cos\theta$, where $0 < d < r$.

8 A point on the circumference of a coin rolling around another coin:



Called a **cardioid** (means "heart-shaped", like "cardiac").
The polar equation is $r = 2a(1 \pm \cos\theta)$.

9 Pencil and string: an **ellipse**.

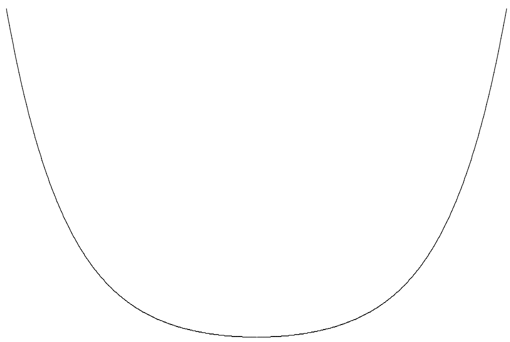


The parametric equations are $x = a\cos\theta$ and $y = b\sin\theta$,

and the Cartesian equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

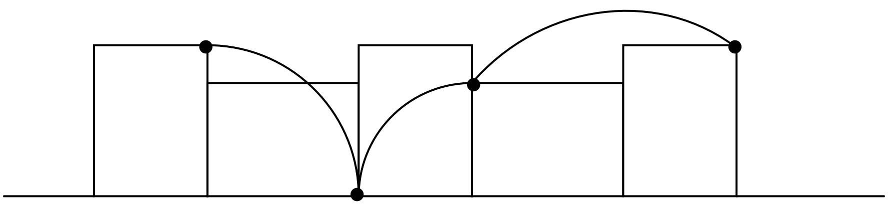
An alternative definition of an ellipse is "the locus of a point which moves so that the ratio of its distance from a fixed point to its perpendicular distance from a fixed line is a constant <1". The fixed point is called the *focus* and the fixed line the *directrix*.

10 Rope suspended between two points:



Called a **catenary**.
Looks a bit like a parabola but it isn't.
The equation is $y = a\cosh\left(\frac{x}{a}\right)$.

11 Corner of book balanced on its end as it "rolls" along the table:
(It helps to draw in the positions of the book at each stage.)



The book is moving from left to right. After it rotates 90°, the next rotation is about the dot, so this point doesn't move. Notice that in the final 90° rotation the dot moves *above* the height of its final position.

The radii of the arcs are 20 cm, 15 cm and $\sqrt{15^2 + 20^2} = 25$ cm (Pythagoras' Theorem, a 5 times enlargement in cm of a 3-4-5 triangle).

12 Mid-point of ladder as it slides (not tips) from against a vertical wall until it is horizontal.
The locus is a quarter of the circumference of a circlem radius half the length of the ladder.

Triangle Properties and Words

name	definition	properties
incentre	point where the 3 ***angle bisectors*** intersect	the ***incentre*** is the centre of the ***inscribed circle***, which touches each of the sides of the triangle
orthocentre	point where the 3 ***altitudes*** intersect (an ***altitude*** is the line joining a vertex to the opposite side so that it is perpendicular to the opposite side)	if you join together the feet of the ***altitudes***, they make another triangle called the ***pedal triangle***, and the ***orthocentre*** is the ***incentre*** of this ***pedal triangle***
circumcentre	point where the 3 ***perpendicular bisectors*** of the triangle intersect	the ***circumcentre*** is the centre of the ***circumscribed circle***, which goes through all 3 vertices
centroid	point where the 3 ***medians*** intersect (a ***median*** is the line joining a vertex to the mid-point of the opposite side)	if the triangle were a thin uniform lamina, the ***centroid*** would be the position of the centre of mass; the ***centroid*** divides the ***medians*** in the ratio 1:2

The orthocentre, circumcentre and centroid are collinear (Euler's line).

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2.9 3-D Solids and Nets

- You need to decide whether or not to use the words "shape" (2-d) and "solid" (3-d) interchangeably. It can be helpful to avoid saying "shape" when referring to a 3-d object.
  Pupils will say "square" when they mean "cube", and you can say, "a square is the *shape* on the *end* of a cube, but what's the whole *solid* called?" (Note that in common usage, "solid" means "hard", so that a pile of sand or a sponge might not be considered solid (or not very), although in science they would be. Also liquids and gases are "solids" in maths!)
- Collect Easter egg boxes: often get isosceles trapezoidal prisms and the occasional pyramid (sometimes truncated). Easter holiday homework can be to look for unusual boxes and bring them in to be named! At other times of the year, chocolate boxes are often interesting solids.
- This can be an encouraging topic for some pupils who often find maths hard, because it relies on quite different skills (e.g., spatial awareness) from those needed in some other areas of maths.

2.9.1 Naming Solids. It's very useful to have actual 3-d solids (cardboard boxes or plastic solids) to pass around the room. "What has David got? Where do you come across triangular prisms?", etc.
Prism: in a certain direction parallel slices are all congruent (e.g., slices of bread) (same cross-section all the way through);
Pyramid: triangular faces that all meet at one point.

(See sheet of drawings, suitable for acetate: point and name: "Can anyone name them all?" Can turn the acetate round and over to vary the appearance.)

A cuboid has 6 rectangular faces: none need be square, or two opposite ones could be square or all 6 could be square, in which case it's a cube.
Cubes, cuboids and cylinders are all prisms.
A triangular prism is a "tent" shape, and a typical glass or Perspex prism in Science will be a triangular prism.
Pencils are sometimes hexagonal prisms and sometimes cylinders (with cones at the point).
A tetrahedron is a triangle-based pyramid. The Egyptian pyramids are square-based pyramids.

"Hold up your solid if you think it's a prism", etc.

2.9.2 I-spy a solid in the classroom; e.g., "I can see a triangular prism" and others have to guess what the object is. Initially give no indication of size. You can also describe mathematically an object (perhaps on the school site) that everyone knows and others have to guess what it is.

e.g., "The object is a hollow cylinder of diameter 8 cm and length 2 cm." Answer: a roll of sticky tape.

2.9.3 Prisms. Instead of just cataloguing solids as prisms or not prisms, you can do it the other way round by asking what solids these could be:
(All the solids are common ones.)

	"cross-sectional shape"	prism?
1	circle	yes
2	circle	no
3	triangle	yes
4	triangle	no
5	square	yes
6	square	no

Answers:
1. *cylinder*
2. *cone or sphere*
3. *triangular prism*
4. *tetrahedron*
5. *cube or cuboid*
6. *square-based pyramid or octahedron*

There are many other possible answers.

2.9.4 What very common everyday object has approximately these dimensions?
20 cm × 10 cm × 8 cm

Answer: ordinary house brick

Could estimate how many used in a house.

2.9.5 Polyhedra. A regular polyhedron (called Platonic) has the same regular polygon for all of its faces, and all its vertices are identical. Find out how many regular polyhedra there are and what they are.

Named after Plato (427-347 BC).

NEED 3-d models or 2-d sketches of regular polyhedra. What is the connection between the numbers of vertices, faces and edges that they have?

Euler's relationship works for convex polyhedra with straight edges and flat faces so long as they don't have "holes" in them!

Answers: There are only 5:

- ***cube** (6 square faces),*
- ***regular tetrahedron** (4 equilateral triangle faces),*
- ***regular octahedron** (8 equilateral triangle faces),*
- ***regular dodecahedron** (12 regular pentagon faces) and*
- ***regular icosahedron** (20 equilateral triangle faces).*

*The regular tetrahedron, the regular octahedron and the regular icosahedron are **deltahedra** (polyhedra whose faces are all equilateral triangles).*

Euler's formula (1707-1783):
vertices + faces = edges +2

2.9.6 What symmetry does a cube have?

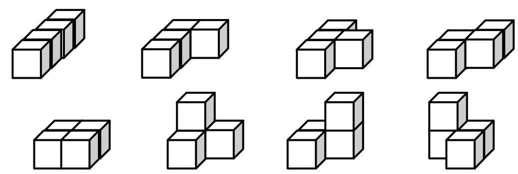
Answer: 9 planes of symmetry (3 parallel to faces and 6 at 45° to pairs of faces); 13 axes of symmetry (3 through the centre of opposite faces, 4 through opposite vertices and 6 through the mid-points of opposite edges).

2.9.7 **NEED** molecular model kit (Science dept.). Chemical molecules and crystals often have symmetrical structures. You can imagine joining every atom to every other atom.
For example, trigonal planar (e.g., BF_3); octagonal (e.g., SF_6); tetrahedral (e.g., CH_4); square-based pyramidal (e.g., IF_5); trigonal bipyramidal (e.g., PF_5); etc.

Some of these molecules have these structures only approximately, and the atoms are always moving around anyway.

2.9.8 **NEED** isometric paper (see sheet) and interlocking cubes. Polycubes.
How many solids can you make by linking together four cubes?
Draw them on isometric paper.

"Isometric" means "equal distance"; the distances from any dot to its six nearest neighbours are all the same.
Must have the paper "portrait" so that there are vertical lines of dots.
The 8 tetracubes are shown below.



Which ones would look different in a mirror (ignore the colours of the cubes)?

Chirality is important with chemical molecules. Different mirror image molecules (enantiomers) have different properties.

Only the last two are "chiral" (have non-superimposable mirror images – like left and right hands). In fact, they are mirror images of each other. All the others have at least one plane of symmetry.

What if I give you another cube so that you have five?

There are 29 so-called "pentacubes", and with more cubes you quickly get vast numbers of polycubes (6 give 166, 7 give 1023, etc.).

2.9.9 **NEED** drawings of "impossible solids" – drawing these on isometric paper (perhaps enlarging them at the same time) is an interesting way of getting used to 3-d isometric drawing.

Suitable for display.

2.9.10 **NEED** "Cube or Not?" sheets, scissors, glue. Pupils should be encouraged to find other nets that will make cubes and to make up puzzles for each other.

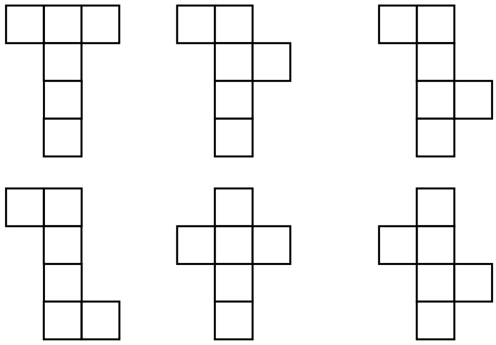
How many possible nets are there for a cube?

Here we count as the same nets which are rotations or reflections (turn it over) of each other.

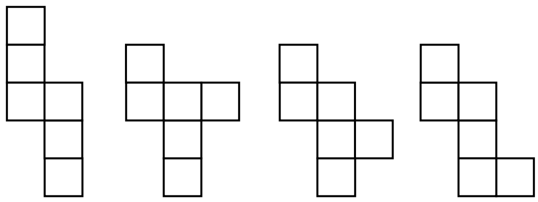
This is a good task to encourage systematic work.

Answers:
A, C, E make cubes; B, D, F don't. ("ACE" is easy to remember as you discuss with pupils.)

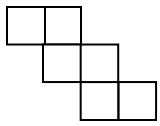
Out of 35 possible hexominoes, only 11 are nets for a cube. Counting systematically:
1. Four squares in a line (6 of these):



2. Only three squares in a line (4 of these):



3. Only two squares in a line (just 1 of these):

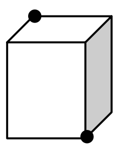


2.9.11 Flaps on a net.
Often we miss them out to keep things simple. If you need flaps for the glue, where do you have to put them?

(See sheets for cube, tetrahedron, triangular prism, octahedron, icosahedron and dodecahedron.)

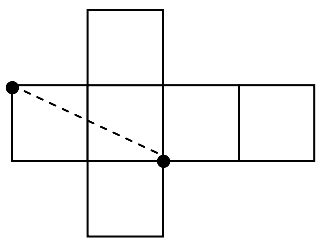
It generally works if you go clockwise or anticlockwise around the perimeter of the net putting a flap on every other edge that you come to.
(If in doubt, it's better to put one, because you can always cut it off if you don't need it but it's more of a problem if you don't have one that you do turn out to need! Cutting off unnecessary flaps is a bit amateurish, though!)

2.9.12 (Need Pythagoras' Theorem)
A spider wants to crawl from the top left back corner of a cube room to the bottom right front corner. If the sides of the room are all 3 m, then what is the shortest distance the spider can crawl?
(No jumping/webbing, etc. allowed!)



You can make up similar puzzles using cuboid rooms, or more complicated solids. The spider can start in the middle of a wall.

Answer: Most people suggest going vertically down the wall and then diagonally across the floor, with a total of $3(1+\sqrt{2}) = 7.24$ *m, but there is a shorter way, most easily seen by drawing the net of the room:*



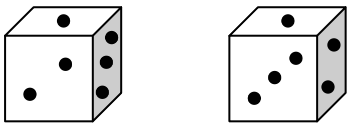
Here the distance is only $3\sqrt{5} = 6.71$ *m.*

2.9.13 What kind of paper would be the easiest to use to draw a net for these solids?

1. a cube or a cuboid;
2. a right-angled triangular prism;
3. a non-right-angled triangular prism;
4. a square-based pyramid;
5. a hexagonal-based pyramid;
6. a tetrahedron;
7. a cone;
8. a sphere.

Answers:
1. *squared (or square dotty) paper;*
2. *if you use a Pythagorean Triple (e.g., 3-4-5), then squared paper is easiest; otherwise, squared paper or plain paper and compasses is best;*
3. *plain paper and compasses or isometric paper if you're careful;*
4. *squared paper;*
5. *isometric paper or plain paper and compasses;*
6. *isometric paper;*
7. *plain paper and compasses;*
8. *no net for a sphere is possible!*

2.9.14 A normal (cube) dice has the numbers 1 to 6 on its six faces and the numbers on every pair of opposite faces add up to 7. Draw a net for such a dice. Are all such dice the same as each other?



Answer: There are two different possible dice like this, sometimes called left-handed and right-handed dice because they are mirror images of each other.

2.9.15 **NEED** acetate of two nets of a cube.
Ask questions like these:

When the net is folded up to make a cube,
1. which faces will be next to the number 1?
2. which face will be opposite the 5?
3. which face will the arrow point to?, etc.

Answers:
1. *2, 4, 6;*
2. *3;*
3. *star.*

2.9.16 **NEED** card, scissors, glue, sticky tape.
Shape Sorter.
This is an object small children play with to get used to matching different shapes/solids. Each solid must fit through one hole only (otherwise the child will get confused!), yet it must go through the right hole reasonably easily (or the child will get frustrated!).

Design and make a shape sorter out of cardboard. Pupils could use a ready-made box (e.g., a shoe box or cereal box) and just cut holes in it and make solids to fit through. Ideally, when all the solids are inside the box the lid will go on for convenient storage.

A challenge is to make a shape sorter that uses only cubes and cuboids.

You cannot have more than one cylinder in your shape sorter. Why not?

The fact that circles have "infinite" rotational symmetry may be why manhole covers are round – there's no risk of the cover falling down the hole. (For example, hexagonal covers on hexagonal holes could do that.)

Pupils tend to be too adventurous at the beginning and need to be encouraged to "practise" by starting with cubes and cuboids before making a twelve-pointed star!

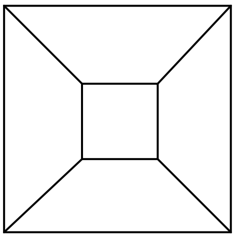
The "holes" need designing first to make sure that none of the solids will go through the wrong holes; e.g., you could begin with a cylinder of diameter 4 cm. A square that won't go through its hole has to have sides longer than $2\sqrt{2} = 2.8$ *cm, but the sides of the square have to be less than 4 cm, otherwise the cylinder will go through its hole.*
At each stage you have to check that nothing you've designed so far will go through the hole for anything else.

If the shapes are prisms, you can neglect the "length" dimension so long as it is long enough not to let the solid go through any of the holes.

Because mathematically similar shapes (all circles are similar) of different sizes will always go through each other (so you can't have more than one cube, either, because all squares are similar).

2.9.17 How many colours do you need to colour the faces of a cube if no two faces that share an edge are allowed to have the same colour? How many different ways can it be done if you have more than this many colours?

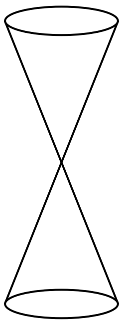
For these purposes, a cube is equivalent to the following 2-d "graph" (the "outside" corresponds to a face as well).



Answer: 3 is the minimum number of colours needed (colour opposite faces the same colour).

no. of colours	*no. of ways*
1	*0*
2	*0*
3	*1*
6	*30*

2.9.18 Conic Sections.
Imagine a double-cone shape, as below, that goes on forever in the up and down directions.

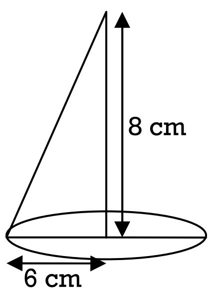


Describe the curves you would get if you sliced through it at different angles with a flat surface (plane).

Answers: Good visualisation exercise.

1. *horizontal plane: a circle (or dot if through the point);*
2. *plane at an angle less than steepness of the sides of the cone: ellipse (or dot if through the point);*
3. *plane at an angle equal to the steepness of the sides of the cone: parabola (or a straight line if through the point);*
4. *plane at an angle steeper than the sides of the cone: hyperbola – two separate bits of curve, discontinuous with asymptotes (or a pair of straight lines if through the point).*

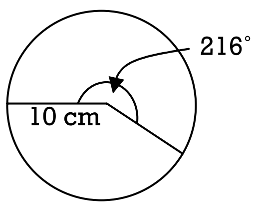
2.9.19 **NEED** A4 plain paper, scissors, sticky tape.
Net of a Cone.
I want to make a cone with a vertical height of 8 cm and a base radius of 6 cm.
What exactly will the net have to be?



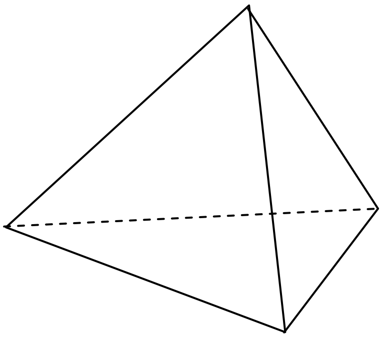
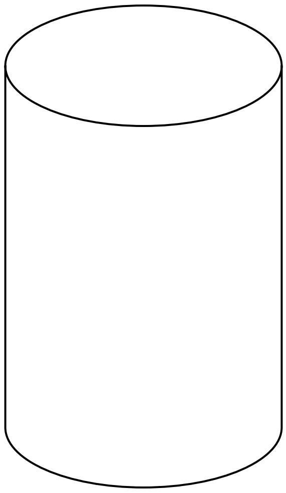
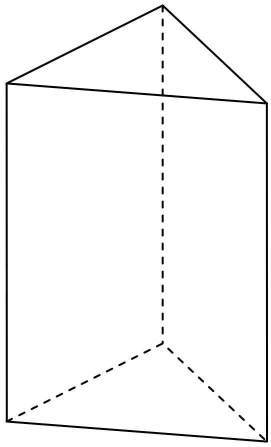
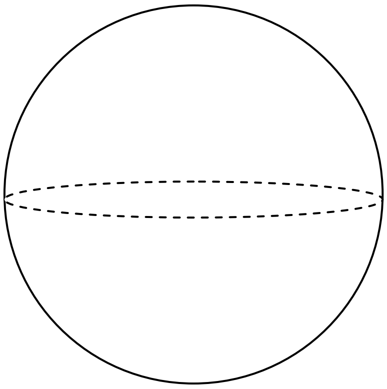
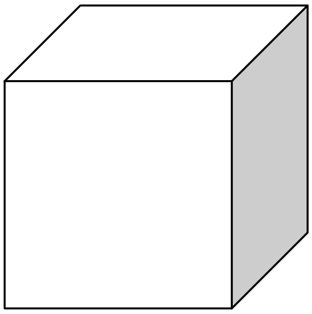
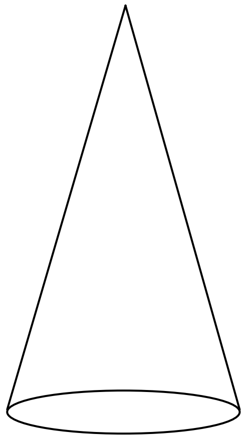
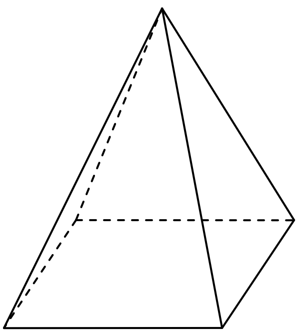
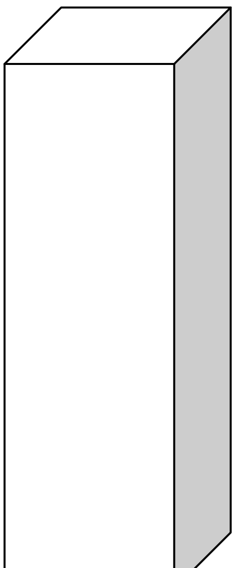
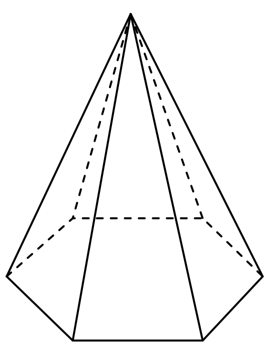
(The cone could be to hold chips or popcorn.)

The cone also needs a circular base of radius 6 cm. Can you cut out everything you need from one sheet of A4 paper?
Yes, if you're careful.

Answer:
You need to cut out a sector of a circle of radius equal to the slant height of the cone. By Pythagoras' Theorem, the sloping sides of the cone will go up $\sqrt{6^2+8^2}=10$ *cm (twice a 3-4-5 triangle). The circumference of the base of the cone will be* $2\pi r = 12\pi$*, so we need the sector of our 10 cm circle to have arc length* 12π*, and this will be* $\frac{12\pi}{20\pi}=\frac{3}{5}$ *of the 10 cm circle's circumference, or* $\frac{3}{5}\times 360 = 216$°*. When cut out and folded up, this will make the required cone.*

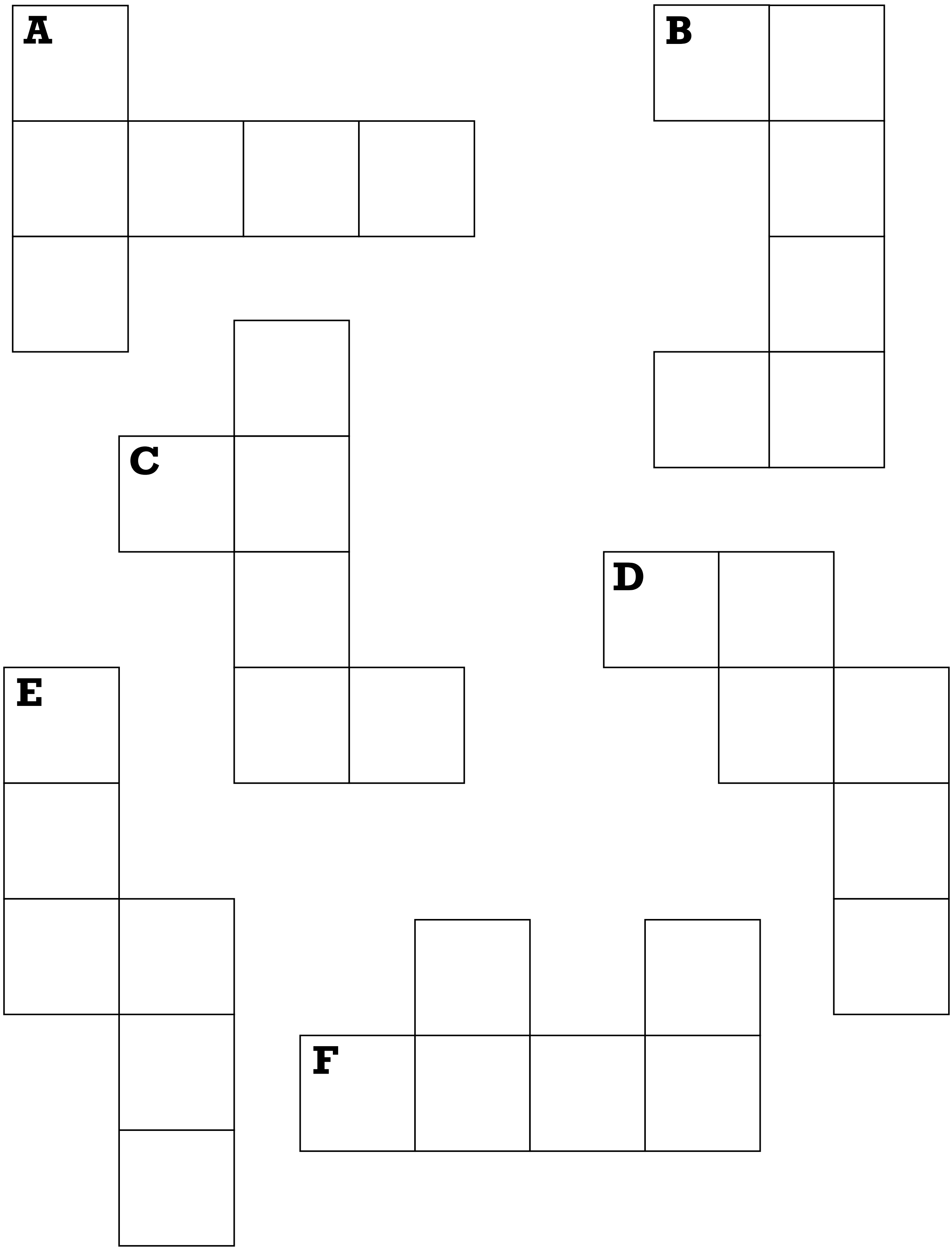


Volume = $\frac{1}{3}\pi r^2 h = 302$ *cm³.*

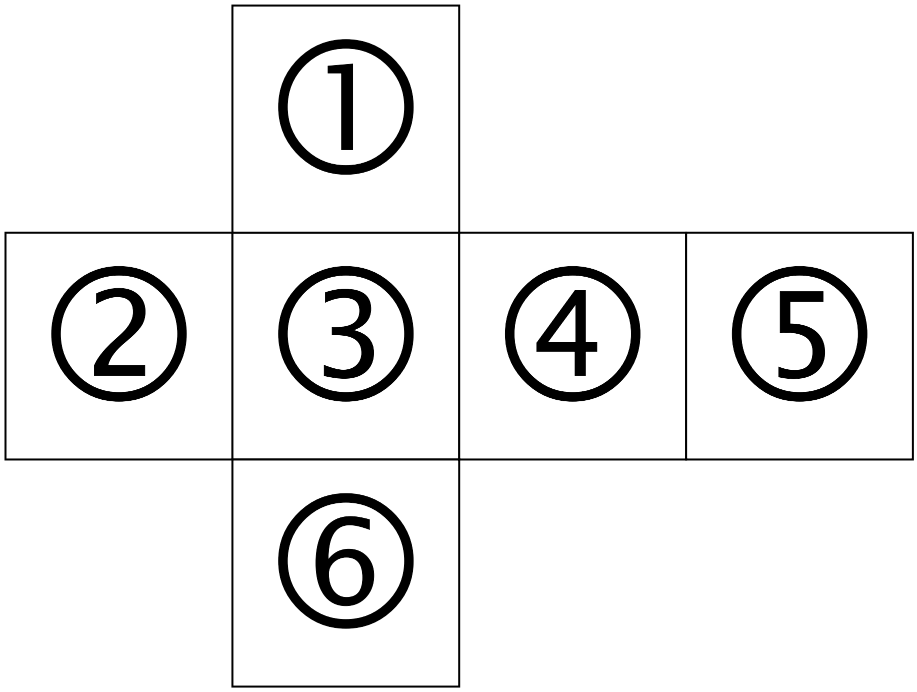










Cube or Not?

Cut out these shapes.
Will they fold up to make cubes?
Try to decide first, then cut them out and see.



Why do some make cubes while others don't?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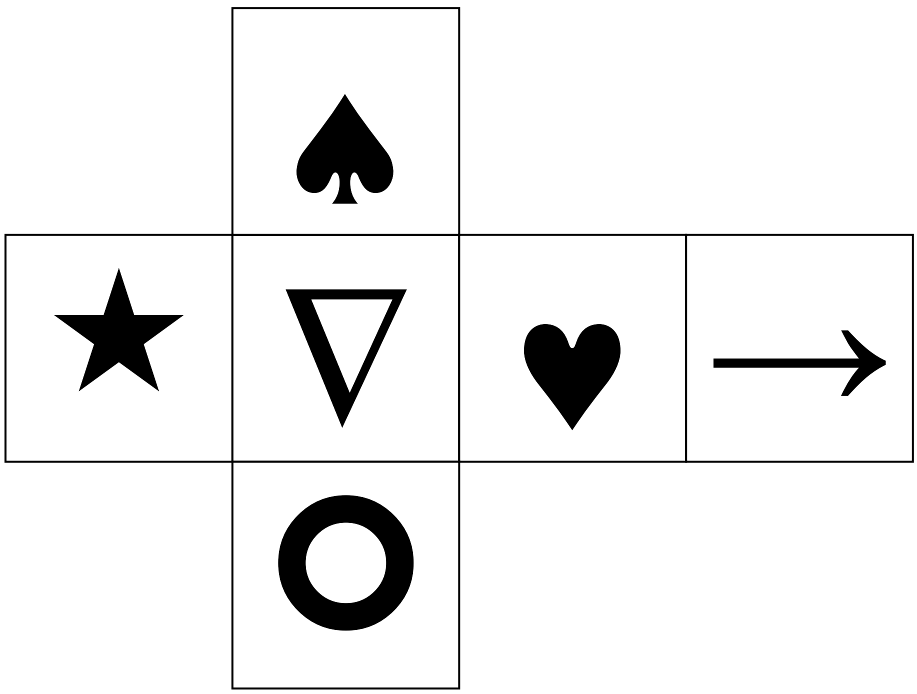
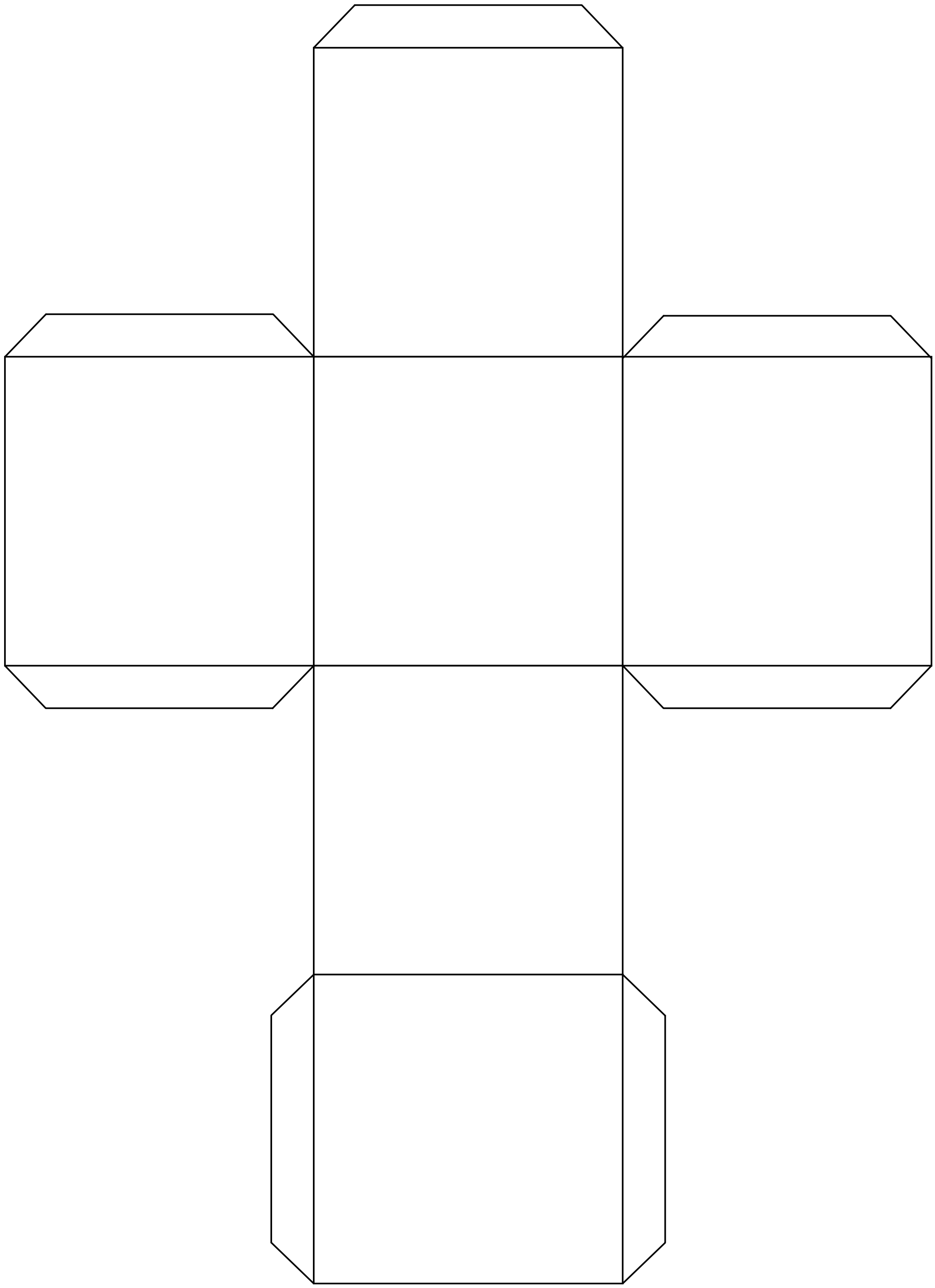


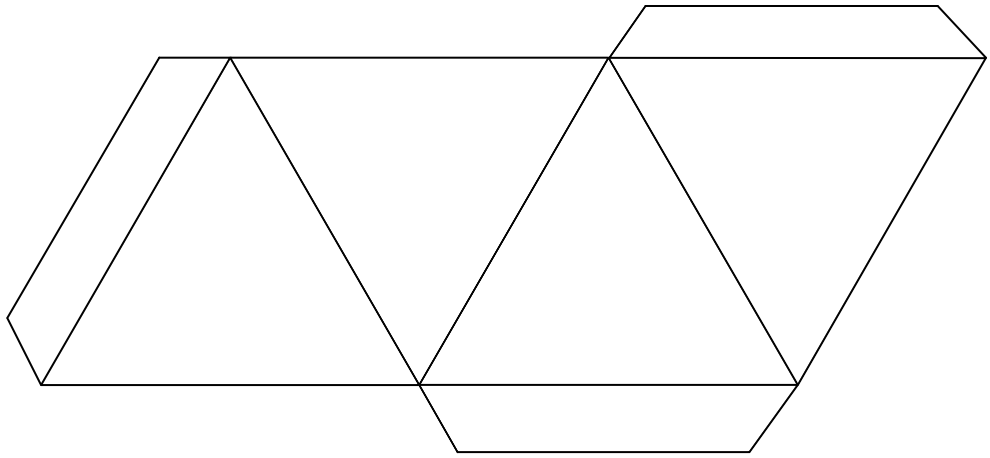
Cube

It's easier to put your name on before you fold it up!



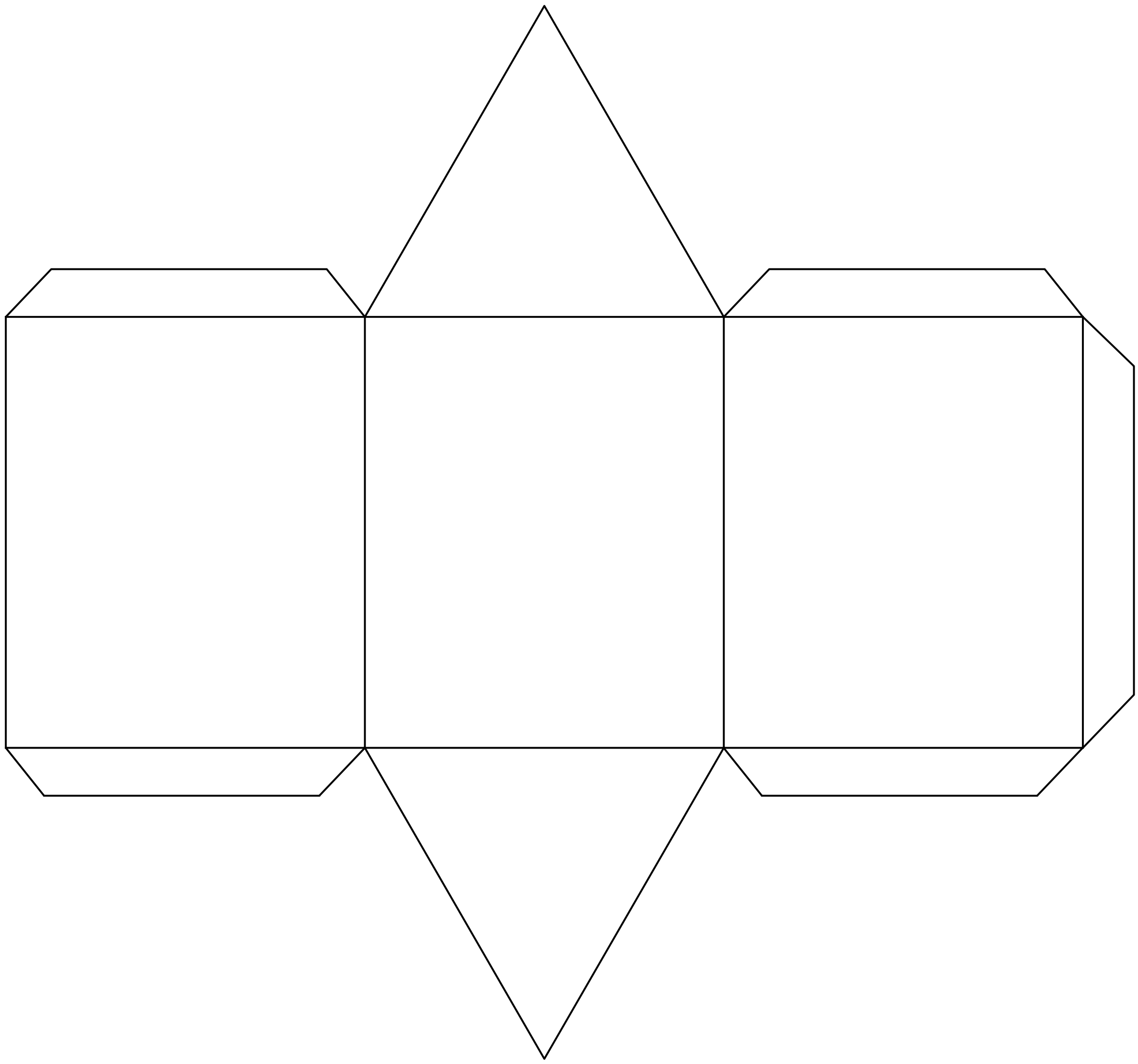
Tetrahedron

It's easier to put your name on before you fold it up!



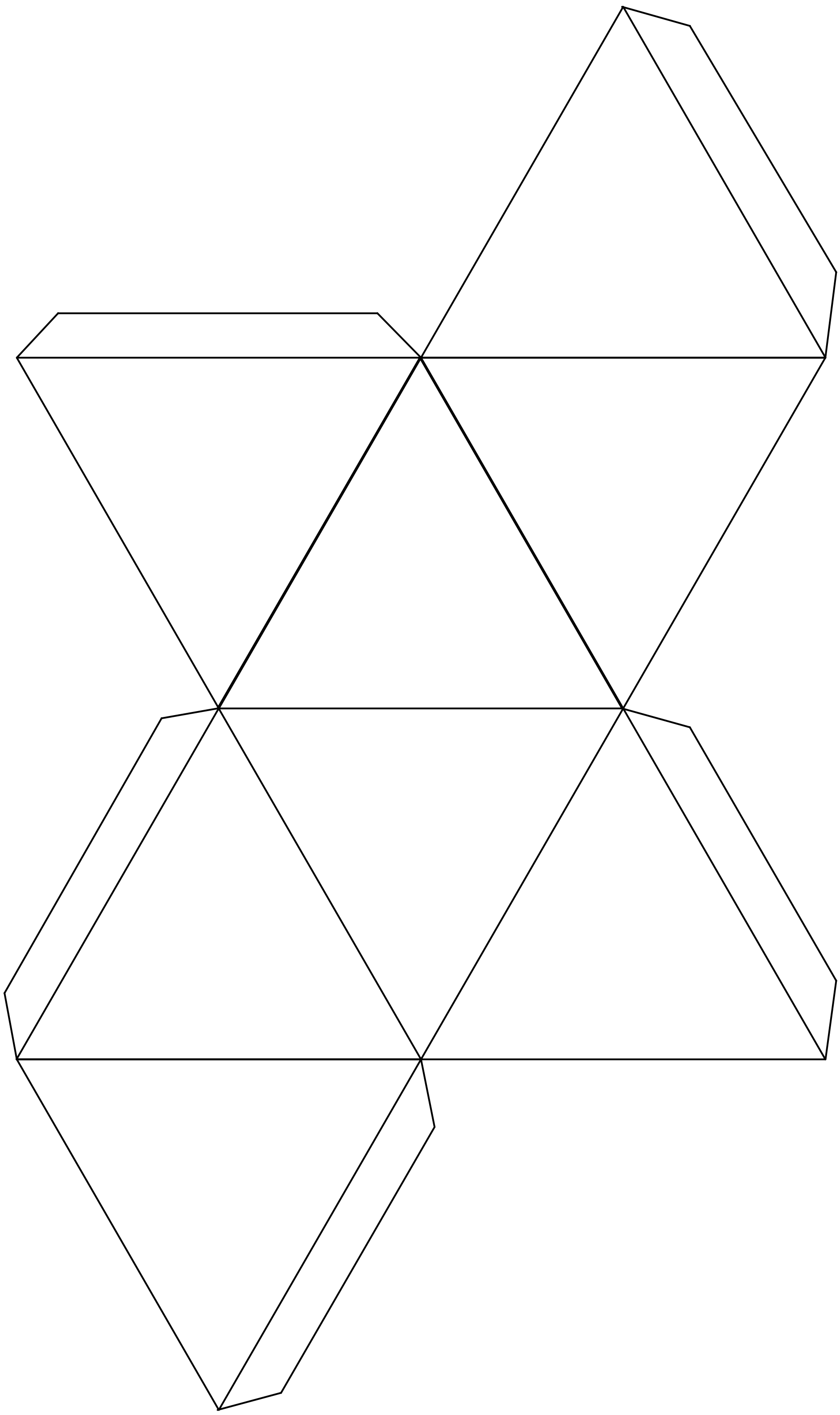
Triangular Prism

It's easier to put your name on before you fold it up!



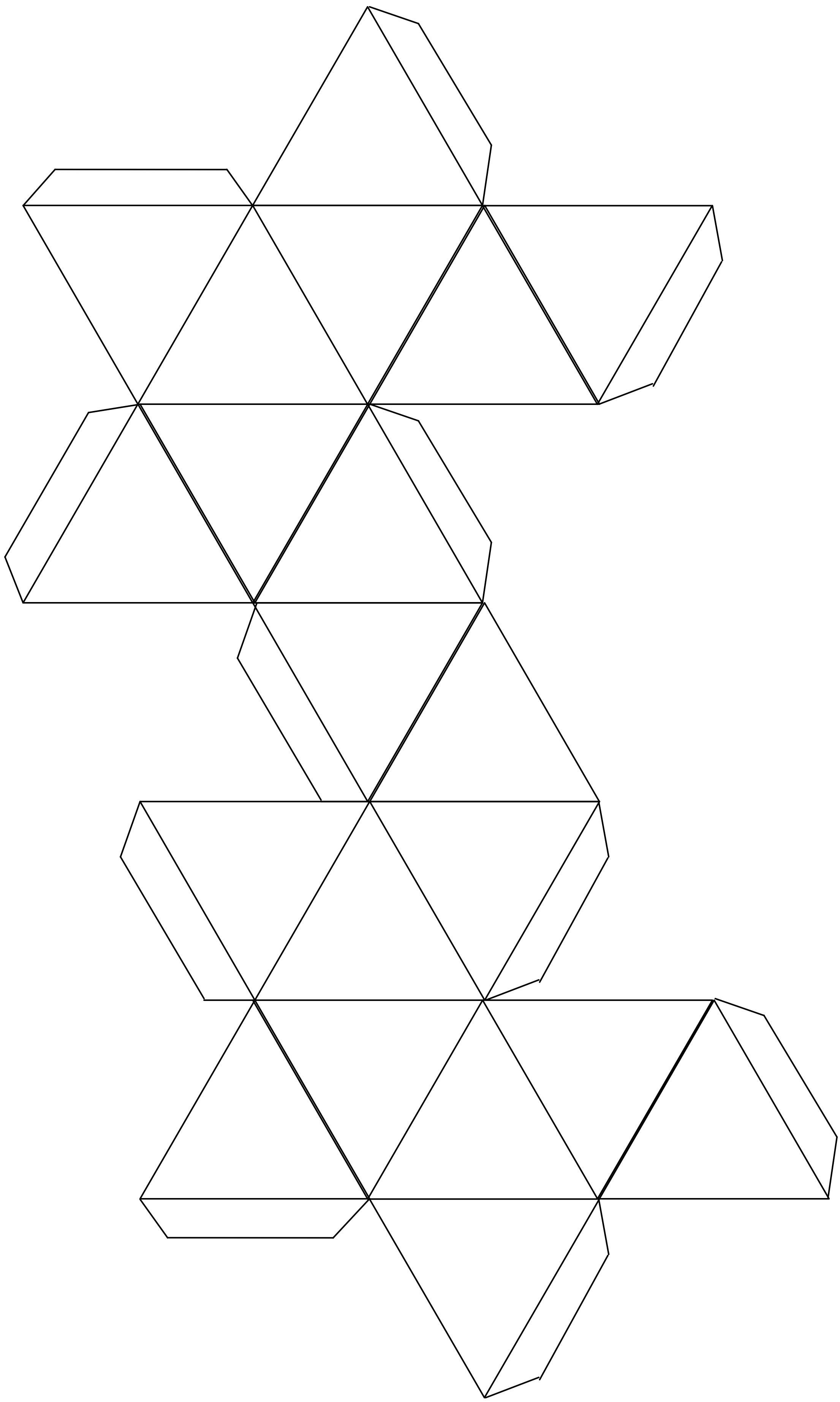
Octahedron

It’s easier to put your name on before you fold it up!



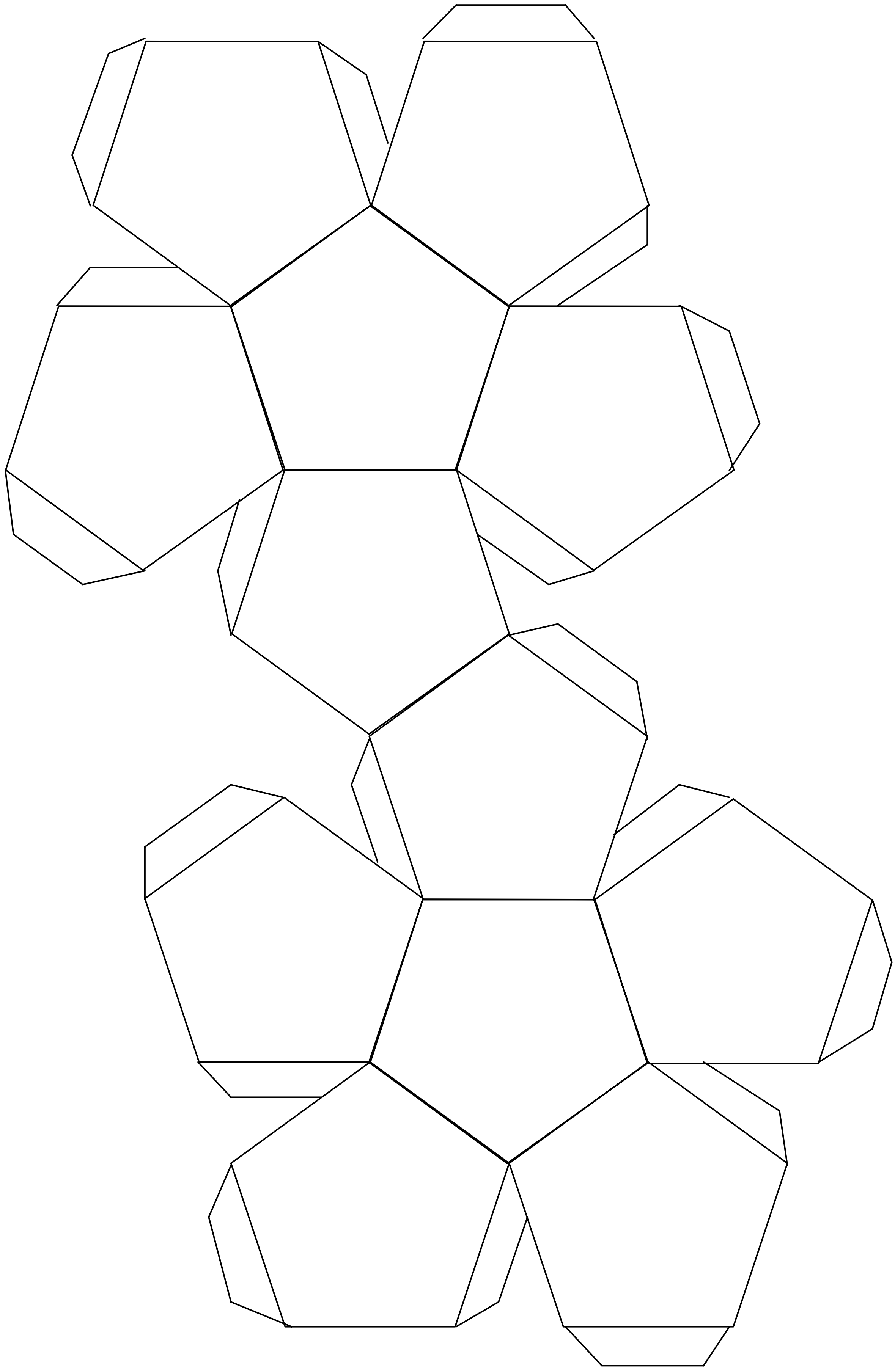
Icosahedron

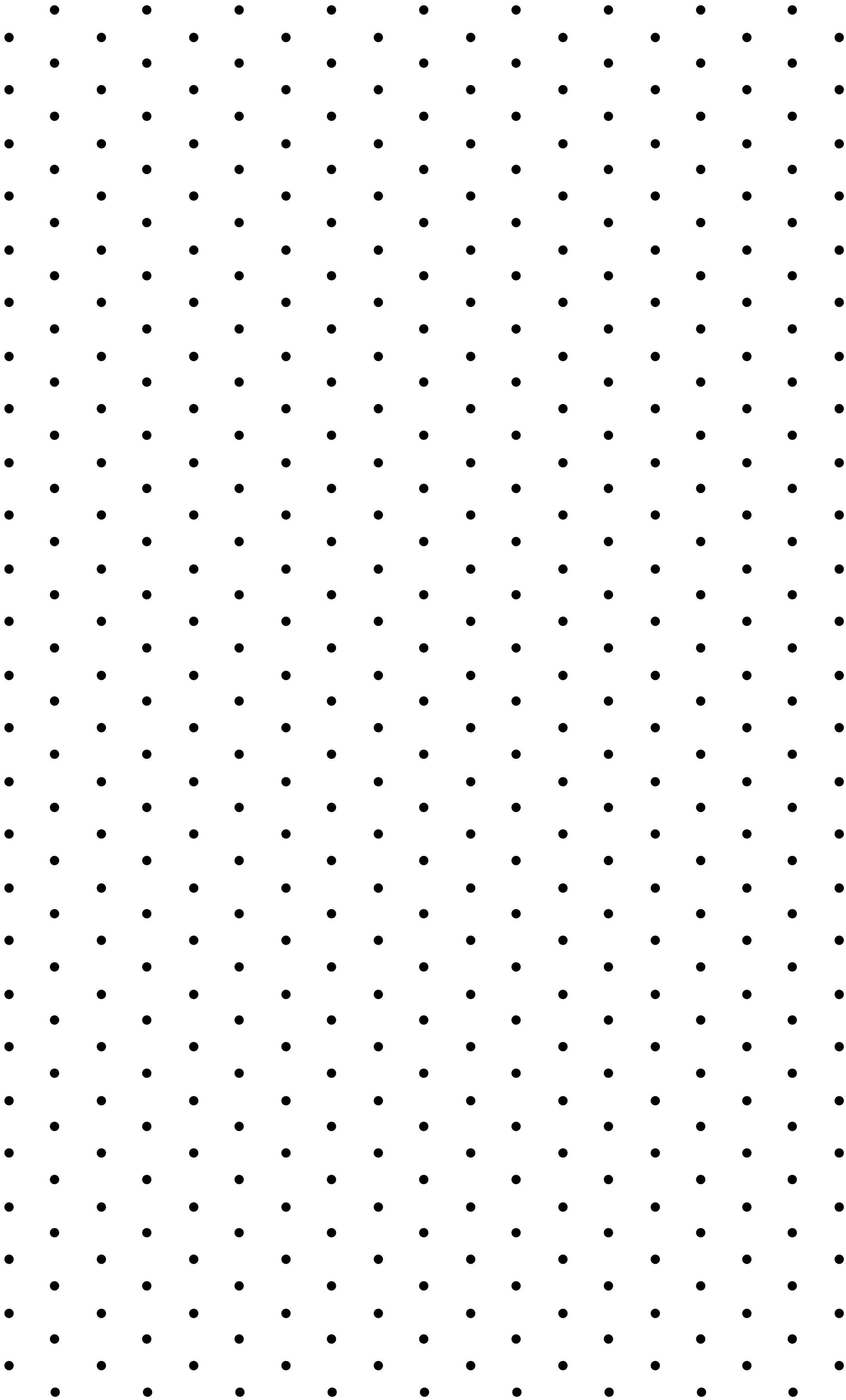
It's easier to put your name on before you fold it up!



Dodecahedron

It's easier to put your name on before you fold it up!





2.10 Volume

- If you're using the words "solid" for 3-d and "shape" for 2-d, pupils need to realise that, of course, you can work out volumes of liquids and gases as well as solids. (In particular, volumes of solids and liquids are more or less constant whereas gas volume depends on pressure, temperature, etc. and not just on the mass.) *Capacity* is just the volume of space (or air) inside a "hollow solid".
- There is some overlap with section 2.15, but the following is pretty essential:

$$\mathbf{cm^3} = \mathbf{ml} \underset{\times 1000}{\overset{\div 1000}{\rightleftarrows}} \mathbf{litre} \underset{\times 1000}{\overset{\div 1000}{\rightleftarrows}} \mathbf{m^3}$$

- Some opportunity to handle and count cubes is essential in the early stages of this topic. Cubes which can be fitted together to make larger cubes/cuboids/etc. are ideal.

2.10.1 Words that mean different things in maths from what they mean in ordinary life or other subjects.
Think of some examples.

In maths/science, volume means how much space something takes up or how much space there is inside something (sometimes called *capacity*).

Take-away, difference, product, factor, prime, negative, positive, sign, odd, even, root, index, power, improper, rounded, interest, expression, identity, solution, term, subject, acute, obtuse, reflex, face, net, square, plane, prism, compasses, translation, sketch, origin, arc, chord, similar, tangent, mean, range, raw, frequency, certain, impossible, independent, etc.
Also volume (loudness in science, vague "amount" in common usage, or can refer to a book).

2.10.2 **NEED** cubes, common cuboid objects.
How many cubes make up this cube/cuboid? You can show 2 × 2 × 2 and 2 × 3 × 4 etc. cuboids to see that volume means the number of cm^3 that will fit inside. Hence multiply the three dimensions to find the volume.

Find the volume, by measuring the dimensions, of common objects: maths book, video cassette, briefcase, locker, room? Start by estimating how many cm^3 would go into it.

If the cubes you're using aren't cm^3 you can say that you're imagining they are.
Stick with integer lengths at this stage.

This is really
volume = area of one layer × number of layers.

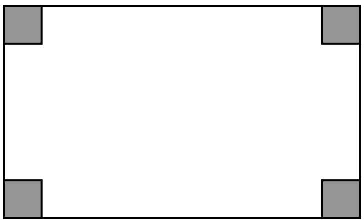
typical values: (pupils tend to underestimate) exercise book: 100 cm^3; textbook: 1000 cm^3 (we're learning a litre of maths this year!) video cassette: 400 cm^3; briefcase: 30 litres; locker: 70 litres (roughly the volume of a human being – some pupils will fit inside their lockers, but don't try it!).

2.10.3 If we woke up tomorrow and everything had doubled in size, would there be any way to tell? (Poincaré, 1854-1912, originally posed this famous riddle.)

More precisely we mean if every length doubled (so 5 cm became 10 cm and so on), because of course that would mean that area had become four times as much and volume eight times as much.

It depends whether other things besides length changed as well. Presumably things would look *the same because our own eyes and bodies would be twice as large (so perspective effects would be the same), but if there were no corresponding increase in mass (for example) it would be easy to detect, because, for example, gravity would be weaker. To make it work, sub-atomic forces would have to increase too.*

2.10.4 **NEED** A4 1 cm × 1 cm squared paper, scissors, sticky tape. Maximum Volume from a piece of paper.
Cut out an 18 cm by 24 cm rectangle from a sheet of A4 paper.
We want to make an open box (no lid) out of this paper that has the maximum possible volume. We'll cut out squares from each corner and see what is the maximum volume we can get.
(Imagine you were collecting sweets in it from a big boxful at the front of the room!)



(diagram not to scale)

Start by cutting off 1 cm × 1 cm squares from each of the four corners. Fold up the sides to make a very shallow box.

Making a cone out of the paper would probably give a smaller volume. (If you cut out a quadrant of radius 18 cm you could roll this up into a cone of slant height 18 cm and base radius $18 \div 4$ = 4.5 cm, so the volume would be about 350 cm^3.)

This size makes for easier calculations than using actual A4 size.

Let x cm be the length of the side of the square cut off. Then we get the following results:

x	*dimensions of box*	*volume (cm^3)*
1	16 × 22 × 1	352
2	14 × 20 × 2	560
3	12 × 18 × 3	648
4	10 × 16 × 4	640
3.4	11.2 × 17.2 × 3.4	655

Could plot a graph of volume against x, but you can see from the numbers that the maximum is between 2 and 4. Trial and improvement gives $x = 3.4$ cm (1 dp).

It's possible to get the same answer using calculus:
$V = x(24-2x)(18-2x) = 4x^3 - 84x^2 + 432x$, *so differentiating,*
$\frac{dV}{dx} = 12x^2 - 168x + 432 = 0$ *for stationary points, and solving this quadratic gives $x = 3.4$ as the only solution in the range $0 < x < 9$.*

2.10.5 **NEED** interlocking cubes.
Minimum Surface Area for a given Volume (the above problem in reverse).
This could be introduced as the problem of wrapping up a number of identical cubes so as to use the minimum amount of wrapping paper. "What's the best shape for a packet of sugar lumps?" would be a more open-ended problem.

The surface area of a solid is the area of its net (excluding any "tabs"), if it has one. (A sphere has a surface area although it has no net.)

"Best" would mean not just the minimum amount of cardboard; you'd have to consider how the packet would look, how easy it would be to fit the design and details on the packet, how stable it would be, etc.

The minimum surface area is obtained when the cubes make a solid that is nearest to a cube in shape (see below).

no. of cubes	***max surface area***	***min surface area***
24	*1 × 1 × 24:* ***98***	*2 × 3 × 4:* ***52***
27	*1 × 1 × 27:* ***110***	*3 × 3 × 3:* ***54***
48	*1 × 1 × 48:* ***194***	*3 × 4 × 4:* ***80***
64	*1 × 1 × 64:* ***258***	*4 × 4 × 4:* ***96***

Maximum surface area comes from arranging the cubes in a long line (a prism with cross-section 1 × 1). In fact, it doesn't affect the surface area if the "line" has bends in it, but then the solid isn't a simple cuboid any more.

In general, for n cubes, the maximum surface area = $4n + 2$.

Think of a situation where maximum or minimum surface area is important.

Keeping warm (huddle up – minimise surface area); getting a sun-tan (spread out – maximise surface area).
Surface tension causes soap bubbles to minimise their surface area (pupils may have seen a demonstration in Science).
Lungs have a very large surface area (over 100 m^2) because that's where oxygen is absorbed.
Granulated sugar dissolves faster than sugar lumps because the water molecules have more exposed sugar to bump into.

2.10.6 Length comparisons versus volume comparisons. People often choose length comparisons to make something seem a lot and volume comparisons to make something not seem that much.

There are issues here of misleading statistics.

How many 10 p coins would you need to make a pile all the way to the top of Mount Everest (8800 m)?
Assume each coin is 1 mm thick.

Answer:
$\frac{8800}{10^{-3}} = 8.8\times10^6$ *coins (= £ 880 000).*

How big a container would you need to put them in?

Each 10 p coin would fit inside a cuboid box 1 mm × 25 mm × 25 mm, which is a volume of $(1\times10^{-3})\times(25\times10^{-3})\times(25\times10^{-3}) = 6.25\times10^{-7}$ *m^3.*
So all of these coins will take up only $(8.8\times10^6)\times(6.25\times10^{-7}) = 5.5$ *m^3; i.e. a cube box with sides 1.75 m (not that big).*

You can do a similar thing with people.
If you lined up all the people in the world end to end, how far would they stretch?
(Assume that they are lying down end-to-end.)

Assuming that there are about 6.5×10^9 *people in the world, and taking an average height of 2 m, the distance would be*
$(6.5\times10^9)\times2 = 1.3\times10^{10}$ *m.*
The average distance to the moon is about 4×10^8 *m, so this is* $\frac{1.3\times10^{10}}{4\times10^8} = 32.5$ *, so they would stretch to the moon and back 16 times.*

What if you put each person in a room 5 m by 5 m by 5 m? How much space would they all take up?

Each room would have a volume of $5\times5\times5 = 125$ *m^3, so for* 6.5×10^9 *people we would need* $(6.5\times10^9)\times125 = 8.1\times10^{11}$ *m^3.*
This is a cube box with sides of length about $\sqrt[3]{8.1\times10^{11}} = 9.3$ *km.*
So a box about 10 km × 10 km × 10 km (not that big) could contain rooms for all the people in the world!

2.10.7 If all the ice in the Antarctic were to melt, how much higher would the oceans rise?

Approximate volume of glacier ice in Antarctica = 3×10^8 km^3;
Approximate ocean surface area = 5×10^9 km^2.

Answer:
If it all melted, the rise would be about
$\frac{3\times10^8}{5\times10^9} = 0.06$ *km = 60 m.*

This ignores many important factors, such as the fact that water is slightly more dense than ice and also that some of the ice is underwater. (When floating ice melts, the water level doesn't change.)

2.10.8 Given that helium has a lifting power of about 1 gram per litre, how many fairground-type balloons do you think it would take to lift an average person?

This value comes from the densities of helium and air. 1 litre of helium has a mass of 0.18 g, whereas 1 litre of air has a mass of 1.28 g. So by Archimedes' principle the difference of about 0.01 N (equivalent to 1 g) is the resultant upward force.

Answer:
We can assume that each balloon is approximately a sphere with a diameter of about 30 cm. Therefore the volume of helium = $\frac{4}{3}\pi r^3 = \frac{4}{3}\pi\times15^3 = 14130$ *cm^3, or about 14 litres.*
So each balloon will lift about 14 g.
An average person of mass 70 kg would therefore need about $70\div0.014 = 5000$ *balloons (rather a lot!).*

2.10.9 Comparing volume and surface area.
What have all these facts got in common?

1. Babies need blankets to keep warm.
2. A mouse can fall a long way and not be harmed.
3. If an ant were enlarged to the size of an elephant it would collapse under its own weight.
(That's why elephants have proportionately wider feet.)

Area scale factor of enlargement = x^2;
Volume scale factor = x^3, and assuming constant density the mass would increase by the same factor.

(Stiletto heels damage some floors.)

Answer:

1. *The amount of heat that human beings can store is roughly proportional to their volume, but the rate at which they lose heat to their surroundings is roughly proportional to their surface area. Being small, babies have a large ratio of surface area to volume.*
2. *The amount of energy the mouse has when it hits the ground is proportional to its mass (or its volume) but the area of impact is proportional to its surface area, so having a lot of surface area for its volume (because it's small) helps.*
3. *If the linear scale factor of enlargement was x, then the ant's weight would be x^3 times bigger, but its legs would be only x^2 times thicker, so they would buckle.*
Pressure on the ground = $\frac{\text{weight}}{\text{surface area}}$.

2.10.10 Design a bucket in the shape of a truncated cone that has a volume (capacity) of 9 litres.

This is a standard 2 gallon bucket.

If the radius at the bottom of the bucket is r, and the radius at the top is a ($r < a$), and the height of the bucket is h, then the volume V is given by $V = \frac{1}{3}\pi h(r^2 + ar + a^2)$.

Notice in this formula that if $a = r$ we obtain $V = \pi r^2 h$, the formula for the volume of a cylinder, and if $a = 0$ we obtain $V = \frac{1}{3}\pi r^2 h$, the formula for the volume of a cone, as we should.

Using values $r = 9$ cm, $a = 12$ cm and $h = 26$ cm gives V just over 9000 cm^3, so this would hold 9 litres. (Many other possibilities.)

Why do you think buckets are not usually cylinders?

Truncated cones will stack inside one another, are stable and are easy to reach inside and clean.

2.10.11 Archimedes (287-212 BC) said that if you put a sphere inside the smallest cylinder that it will just fit into, the volume of the sphere is $\frac{2}{3}$ of the volume of the cylinder.
Can you prove that he was right?

Answers:
Let r be the radius of the sphere. Then the height of the cylinder will be $2r$, so volume of cylinder = $\pi r^2 \times 2r$ = $2\pi r^3$ and volume of sphere = $\frac{4}{3}\pi r^3$ (standard result) which is $\frac{2}{3}$ of $2\pi r^3$.

What is the relationship between the two surface areas? (Assume that the cylinder has open ends.)

surface area of cylinder = $2\pi r \times 2r = 4\pi r^2$ and surface area of sphere = $4\pi r^2$ (standard result), so they're equal.

What if the ends are closed instead?

Then, surface area of cylinder = $4\pi r^2 + 2\pi r^2 = 6\pi r^2$; i.e., 50% more than the surface area of the sphere.

2.10.12 Archimedes' Principle.
Why do some things float and others don't?

Small insects and objects can sit on the surface of water because of surface tension, and that is a different phenomenon – they're not really "floating".

Whether something will float depends both on its mass (or weight) and on its shape.
As an object sinks into the water, the water pushes upwards on it and the force upwards is equal to the weight of the water the object has displaced. If the object can displace water with as much weight as the total weight of the object before it is completely submerged then it will float.

This will happen only if the average density of the object is less than the density of water (1 g/cm^3).

2.10.13 Inside Faces.
If you make a 3 × 4 × 5 cuboid from 1 × 1 × 1 cubes, how many faces of the cubes *can't* you see? (You're allowed to turn the cuboid around to look at it.)
Start with a 1 × 1 line of cubes and build up gradually.

Answer:
Imagine an $x \times y \times z$ *cuboid where* x, y *and* z *are all positive integers.*
Since each cube has 6 faces, altogether there are $6xyz$ *faces. On the outside are* $2xy$ *visible faces from one pair of parallel faces, and* $2xz$ *and* $2yz$ *from the other two pairs of parallel faces. So the total number of inside faces must be* $6xyz - 2xy - 2xz - 2yz$.

What if the cuboid is standing on a table, so that you can't see the faces underneath either?

If it were one of the xz *or* yz *faces that was standing on the table, then it would be the co-efficient of those terms that would change from –2 to –1.*

In this case, say it's one of the $x \times y$ *faces that is standing on the table. Then you just lose sight of* xy *faces, so the total number of unseeable faces increases to* $6xyz - xy - 2xz - 2yz$.
If $y = z = 1$, *then total =* $3x - 2$, *for example.*

2.10.14 **NEED** tape measures, possibly other things as well. Estimate the volume of a human being.

Answer: the value is not important; it's the process adopted that matters.

Practical methods: could be done at home as a homework; e.g., mark side of bath before and after getting in (use something that will rub off!). Measure the difference in height and multiply by the cross-sectional area of the bath.
Theoretical methods: e.g., ignore hands, feet, etc., and treat the human body as a sphere on top of a cuboid with two identical cylindrical arms and two bigger identical cylindrical legs.

Size will obviously depend on age of pupils.

Theoretical approximation:
Head: $\frac{4}{3}\pi r^3 = \frac{4}{3}\pi 10^3$ *= 4.2 litres;*
Trunk: 20 × 50 × 50 = 50 litres;
Arms: 2 × $\pi r^2 l$ *= 2 × 3.14 ×* 4^2 *× 50 = 5 litres;*
Legs: 2 × $\pi r^2 l$ *= 2 × 3.14 ×* 6^2 *× 80 = 18 litres;*
So total estimate = 77 litres approx, which seems sensible.

(See similar task in section 2.2.17.)

If you were flattened by a steamroller so that you were only 5 mm thick, how big a splat would you make?!

Area = Volume/height = 0.08/0.005 = 16 m^2*;*
i.e., a 4 m by 4 m square!
(Be cautious if some pupils may be upset by this!)

2.10.15 Find out the world record for the number of people who have simultaneously fitted inside a standard telephone box.
(Possible homework.)

Answer: About 20, depending on the type of telephone box and the exact rules about whether you have to close the door or be able to use the telephone!

Estimate a theoretical maximum.

Can estimate an average human volume (see above) or estimate by taking average density as 1 kg/litre (the same as water, since we just float) and an average human mass as 70 kg. So our volume is about 70 litres.

The dimensions are about 3 ft × 3 ft × 8 ft, so the total volume = 72 cu ft (= 2 m^3 *approx).*
Assuming the average volume of a human being is 70 litres (see left), we would estimate a maximum of about 2000/70 = about 30 people.
In practice a lot fewer.

2.10.16 When no-one is using it, the water in the swimming pool comes up to 50 cm below the level of the floor outside the pool. How many people would have to get into the pool (completely submerged) to make it overflow? (We'll assume the people are still, not jumping around and making waves!)

Answer: We could assume that the pool is 50 m by 25 m, so the area of the water's surface = 50 × 25 = 1250 m^2*. We need to raise this by 50 cm, so the volume increase needed is 1250 × 0.5 = 625* m^3*. If we take an average human volume as 70 litres, then it would take 625 000/70 = about 9000 people! (Not very practicable!)*

The bottom of the pool actually slopes from one end to the other so that one end is deeper than the other. What difference would it make if we took this into account?

It would make no difference since that extra space will be filled with water throughout.

2.10.17 How many identical packets (cuboids 3 cm × 4 cm × 5 cm) can you fit into these cuboid containers?

1. 30 cm × 40 cm × 50 cm;
2. 30 cm × 40 cm × 51 cm;
3. 30 cm × 40 cm × 52 cm;
4. 30 cm × 40 cm × 53 cm;
5. 30 cm × 40 cm × 54 cm.

When the answers to these divisions are not integers, you always need to round down.

Try these ones (same size packets):

1. 10 cm × 15 cm × 20 cm;
2. 10 cm × 10 cm × 15 cm;
3. 10 cm × 10 cm × 10 cm;
4. 10 cm × 11 cm × 12 cm.

Try making up some puzzles like these.

A spreadsheet makes this much easier.

Answers:
1. 1000; 2. 1020; 3. 1040, 4. 1040 (still), 5. 1080.
Provided the packets fill the entire container with no empty space, you can divide the volumes; i.e., for question 1, $\frac{30\times40\times50}{3\times4\times5}=1000$*, but a safer way (and necessary if there are going to be any gaps) is to think how many rows you'll get along each dimension; i.e.,* $\frac{30}{3}=10$ *along the 30 cm side,* $\frac{40}{4}=10$ *along the 40 cm side and* $\frac{50}{5}=10$ *along the 50 cm side, and*
10 × 10 × 10 = 1000.

1. 50; 2. 20; 3. 12; 4. 18.

In general, if the sides of the container have lengths A, B *and* C*, and the sides of the packets have lengths* a, b *and* c*, you need to work out the six products* $\frac{A}{a}\frac{B}{b}\frac{C}{c}$, $\frac{A}{a}\frac{B}{c}\frac{C}{b}$, $\frac{A}{b}\frac{B}{a}\frac{C}{c}$, $\frac{A}{b}\frac{B}{c}\frac{C}{a}$, $\frac{A}{c}\frac{B}{a}\frac{C}{b}$*, and* $\frac{A}{c}\frac{B}{b}\frac{C}{a}$*, in each case doing "integer division" (normal division but rounding down and discarding the remainder). You have to see which of these six products gives the maximum number of packets.*

2.10.18 How long is a toilet roll?
You want to know how much paper there is on a toilet roll without unrolling the whole thing. What measurements could you take?

Answers:
One option would be to weigh the roll and to weigh one sheet and do a division. (It would be more accurate to count off 20 sheets, say, to weigh and then divide by 20.) To do this, you would need an accurate balance and you would have to weigh a cardboard tube separately and subtract this from the total. You would calculate how many sheets were on the roll and then multiply this by the length of one sheet.

A second option would be to measure the thickness *of one sheet (again, you would measure 20, say, and divide by 20) and the thickness of the roll, and divide to find out how many layers there are on the roll.*
This number can be multiplied by the average circumference $\frac{1}{2}(R+r)$*, where* R *is the outer radius and* r *is the radius of the cardboard tube.*
Since the thickness of paper on the roll is $(R-r)$*, and if* t *is the thickness of one sheet, then the number of layers on the roll is* $\frac{R-r}{t}$ *so the length of the roll is* $\frac{R-r}{t}\times2\pi\left(\frac{R+r}{2}\right)$

$$=\frac{\pi(R^2-r^2)}{t}.$$

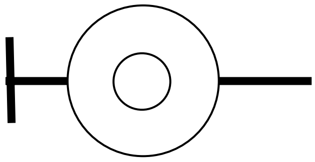
Another way of arriving at this formula is to think about the area of the end of the roll (the cross-sectional area), which is $\pi R^2-\pi r^2$ $=\pi(R^2-r^2)$*, and this will be the same as the thickness of one sheet multiplied by the length of the whole roll.*

So length of the roll is $\frac{\pi(R^2-r^2)}{t}$ *again.*

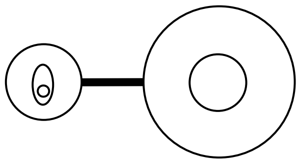
2.11 Plans and Elevations

- If you can lay your hands on some real architect plans/blueprints, that would show the relevance of this topic. Or pupils could bring in diagrams from instructions for putting together some object such as a bookcase, a climbing-frame or a model. Notice how hard it would be without the diagrams.
- Pupils could look at home for optical illusions that depend on different points of view. Perhaps there are artists who have exploited views of common objects from unusual angles in their work.
- You may be able to find satellite photographs and aerial photographs on the internet – perhaps of the local area.
- Pupils with experience of playing certain types of computer games may have an advantage with this topic!

2.11.1 You can start with "A Mexican on a bicycle",



and "A Mexican frying an egg"!



If pupils have access to a camera they could photograph "common objects" viewed from unusual angles to produce a set of puzzles. (For each object you also want a view from a more usual *angle to use as an "answer".)*

Skip the Mexican idea if it might offend someone.

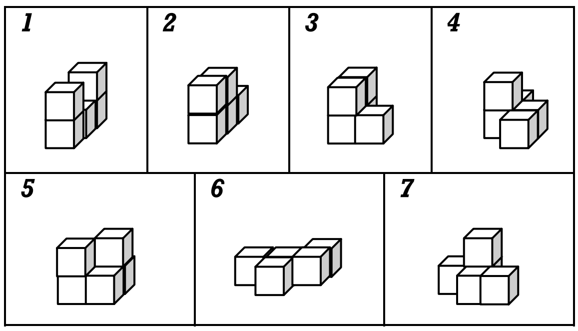
Pupils may know of other drawings like this.

2.11.2 **NEED** interlocking cubes, "What are these objects?" sheets.

Some pupils (and teachers!) find this sort of task very hard.

Objects with planes of symmetry (how many in brackets): 1 (2); 2 (1); 3 (1); 5 (1); 6 (1). Objects 4 and 7 are the only "chiral" objects among these; they are non-superimposable mirror images of each other (like enantiomers in Chemistry).

Answers: There are 29 pentacubes altogether. The 7 used are shown below.



2.11.3 Escher (1898-1972) drawings are very impressive to look at.
Pupils could attempt some "impossible drawings" on isometric paper.

Many books have suggestions of impossible drawings.

2.11.4 Scale Drawings.
This topic could be tied in to work on scale drawing by making "architect's plans" for a room or floor at school, at home or elsewhere.

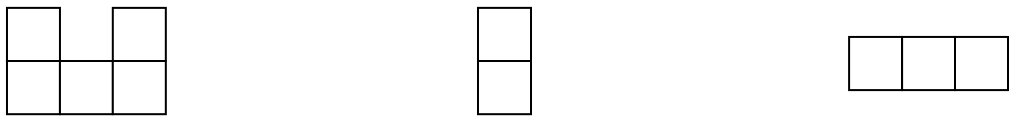
What are these Objects?

All of these objects are made out of 5 interlocking cubes.

There are three different views of each object.
Try to make the object and draw it on isometric paper.

front view | ***left side view*** | ***plan view***

1



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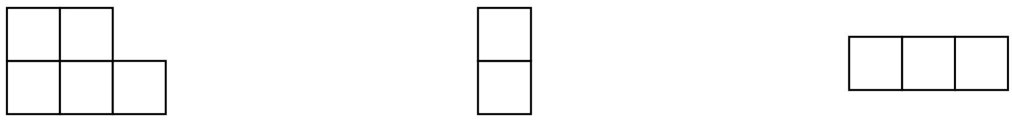

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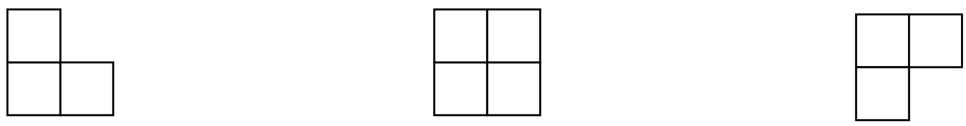

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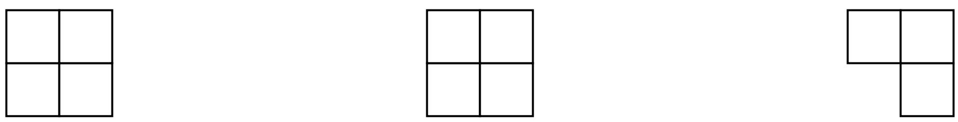

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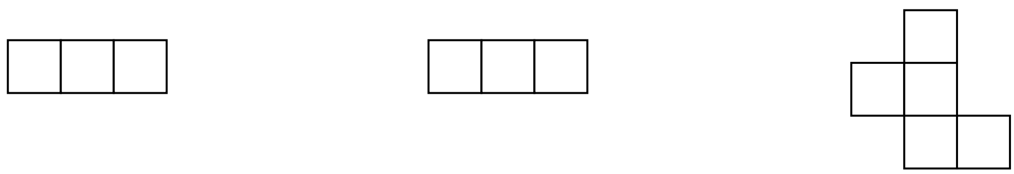
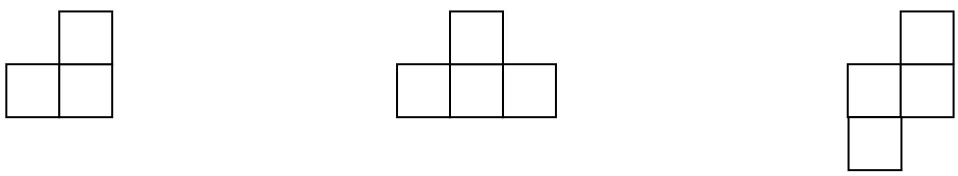


7



Which objects have a plane of symmetry?
Which two objects are mirror images of each other?

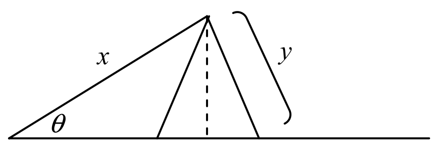
2.12 Similarity and Congruence

- These terms can apply to plane shapes or to solids: *similar* means that one shape/solid is an enlargement of the other; *congruent* means that the shapes/solids are identical (a special case of similarity with scale factor 1). Reflections (turning the shape over) count as congruent.
  Pupils may want to use the equals sign to indicate congruence (e.g., $\Delta ABC = \Delta PQR$), and this might be an opportunity to discuss whether = always means the same thing in maths or not.
- For material on similarity, see section 2.13.

2.12.1 Two triangles will be congruent if one of these conditions is true.

1. the three sides of one of the triangles are the same as the three sides of the other (SSS);
2. two sides and the angle in between are the same (SAS);
3. two angles and the side in between are the same (ASA);
4. the triangles are right-angled and the hypotenuse and one other side match (RHS).

Why are these the only conditions that guarantee congruence?



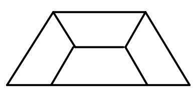
There is much logical thinking involved here.

If all three angles match (AAA) then the triangles must be similar and might be congruent but needn't be.

If two sides match (SS), the third side can be anything between the sum of the two given sides and their difference, so there are infinitely many possibilities.
One side and any two angles (SAA) is equivalent to ASA because in a triangle the angles must add up to 180°, so given two angles you can always work out what the third must be, but the angles and sides must correspond.

The crucial case to think through is ASS (the angle isn't *between the two given sides).*
Here, what happens depends on how long the second side is relative to the first.
If the angle is θ, the first side has length x and the second side has length y, then we get the diagram on the left.
If $y < x\sin\theta$, the sides don't join up and there is no such triangle.
If $y > x\sin\theta$, then there are two possible triangles depending on which side of the vertical dashed line the third side goes.
If $y = x\sin\theta$, then there is just one possible triangle and it's right-angled (this is the RHS possibility mentioned already).

2.12.2 Find some shapes that can be cut up into two or more pieces which are all mathematically similar to the original shape.
Start with triangles.



Answers: (number of similar shapes produced in brackets)

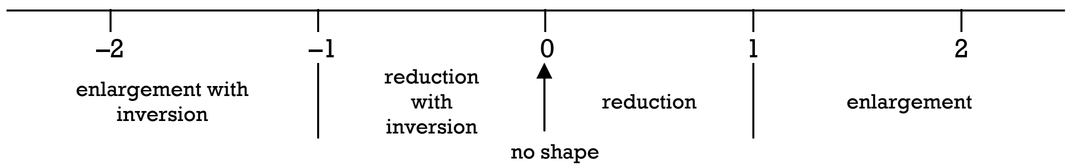
- *any right-angled isos. triangle (2, 3, 4, …);*
- *any equilateral triangle (4, 7, 9, 10, …);*
- *any parall'm, including rectangles, with sides in the ratio $1:\sqrt{2}$ (2) (like A-size paper);*
- *any parall'm at all, including rhombuses, rectangles and squares (4, 9, 16, …);*
- *special trapeziums, e.g., see left (4)*
- *lots more possibilities*

2.13 Symmetry

- Pupils need to be able to recognise and describe fully a transformation that's already happened. Certain information is needed to describe a transformation fully:

reflection	**rotation**	**translation**	**enlargement**
where is the mirror line? *give its equation if on co-ordinate axes*	where is the centre of rotation? what is the angle of rotation? what is the sense of the rotation? *positive (anti-clockwise) or negative (clockwise)*	how far and in what direction has the shape moved? *give the vector* $\begin{pmatrix}\text{distance to the right}\\ \text{distance up}\end{pmatrix}$	where is the centre of enlargement? what is the scale factor?

For enlargement you can draw a scale factor number line:



A scale factor of 1 leaves the shape unchanged; a scale factor of –1 inverts the shape but doesn't change its area.

- Pupils need to be able to perform a specified transformation of a shape. It's good to develop a culture where pupils check their own drawings by measuring lengths and angles so that they rarely need to ask "is this right?" Accuracy of ± 1 mm and ± 1° should be the aim.
- Marking is much easier when drawings are done on co-ordinate axes. Pupils can then write down the co-ordinates of the points of the image shape(s) and it is easy to see whether these are correct or not. Otherwise the only quick way to mark drawings is to do the accurate drawing yourself, photocopy onto an acetate (this is much easier – although more expensive - than trying to do an accurate drawing using acetate pens) and then place this over the pupil's work.
  You can set up some kind of rule for marks; e.g., within 1° or 1 mm, 2 marks; within 2° or 2 mm, 1 mark.
- Coloured acetate is available from art shops, and although it's expensive you can do a lot with just one sheet. You can cut out various shapes and place onto an acetate of 1 cm × 1 cm squares (available in section 1.23). This is particularly helpful with visualising translations. Dynamic Geometry software can make this even slicker.
- Small mirrors can be useful, and tracing paper is more or less essential for many pupils.

2.13.1 Reflections.
Completing the other half of pictures of animals/aliens, etc. can be useful practice.

Good for displays.

The reflected image is always *congruent* to the original object.

Diagonal mirror lines are sometimes easier to do by rotating the paper so that the line goes away from you. You need to count the squares (or diagonals of squares) or measure in a direction perpendicular to the mirror line. "Diagonal" mirror lines that are not at 45° are very difficult to do accurately unless the shape is well chosen.

What stays the same and what changes in a reflection?

Same: size, shape, lengths of sides, area, angles;
Different: position, "orientation", "handedness".

2.13.2 **NEED** "Symmetrical Squares" sheets.

Several possible answers.

2.13.3 Introduce by drawing axes from –6 to 6 in both directions on the board.
Plot the co-ordinates A(1,1), B(1,4), C(2,4), D(2,2), E(3,2) and F(3,1) and join them up to get an L-shape.

Recaps plotting co-ordinates.

Or you can use a scalene right-angled triangle. You don't want to use anything with symmetry because that sometimes makes it hard to see if the shape has been changed or not, although so long as the vertices are labelled clearly this does not have to be a problem.

I'm going to add 3 to all the coordinates to get six new points. So A becomes (4,4). What do you think the new shape will be like?
We're adding 3 to the x-number (the first number) and 3 to the y-number (the second number).

Many will think shape will be stretched or enlarged.

Actually it's just a translation $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$.

So transforming the co-ordinates (x, y) *into* $(x+3, y+3)$ *is the translation* $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$.

What if leave the x-numbers alone and make the y-numbers into *minus* what they are?
i.e., $(x, y) \rightarrow (x, -y)$

This time it's a reflection in the x*-axis.*

Put up a list of possible co-ordinate transformations. Pupils can invent their own. They could work in groups so as to cover all these as a class in a reasonable amount of time.
Make a table of results.

$(x, y) \rightarrow$	***transformation***
$(x+3, y+3)$	*translation 3 to the right, 3 up*
$(x, -y)$	*reflection in* $y = 0$
$(-x, y)$	*reflection in* $x = 0$
$(-x, -y)$	*rotation 180° about (0,0)*
(y, x)	*reflection in* $y = x$
$(y, -x)$	*rotation –90° about (0,0)*
$(-y, x)$	*rotation +90° about (0,0)*
$(-y, -x)$	*reflection in* $y = -x$
$(x+1, y-3)$	*translation* $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$
$(2x, 2y)$	*enlargement, scale factor 2, centre (0,0)* ***[Need vertical axis up to 8 for this one.]***

Try to generalise; e.g.,
$(x+a, y+b)$ is a translation $\begin{pmatrix} a \\ b \end{pmatrix}$
(even if a or b are negative).

(ax, ay) is an enlargement, scale factor a centred on the origin.

Try out more complicated ones; e.g.,
$(x, y) \rightarrow (3x-2, 3y+1)$.
An enlargement, scale factor 3 about the origin followed by a translation 2 units to the left and 1 unit up.

2.13.4 An alternative approach is to use Dynamic Geometry software to allow pupils to explore different transformations on a shape of their choice and investigate what happens to the co-ordinates of the vertices under each different transformation.

The software may allow other transformations such as stretch and shear.

Some pupils might like to investigate matrices to try to work out the effects of putting different numbers in the four different "slots".

2.13.5 Is a human face symmetrical?

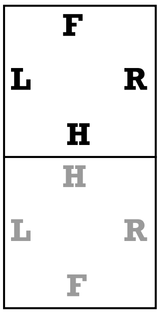
There is more or less a vertical line of symmetry, but not exactly. If it were, our faces would look the same in the mirror, and they don't. Some studies suggest that highly symmetrical faces are the most beautiful.

What's the minimum change you'd have to make to a human face to give it some rotational symmetry?!
(Pupils can sketch their ideas.)

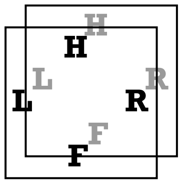
You could aim for order 2 rotational symmetry, and even that is not easy.

2.13.6 Why does a mirror swap round left and right but it doesn't swap round up and down?
I mean why is my left hand where my right hand is (and vice versa), but my head isn't where my feet are (and vice versa)?

Does a mirror really know which way is up? (What we mean by "up" is really something like "the opposite way to gravity" – how could a mirror know about gravity?)

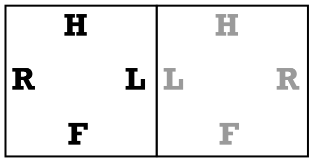


Answer: This is quite a tricky one.
It's really because we imagine our mirror image standing (upright) next to us.
If you're facing a mirror straight-on, every point is reflected exactly in front of the original point.



(Black for the person; grey for the reflection; H= head, F = feet, L = left hand, R = right hand.)

But to compare the original with the image, you have to turn one of them over so you can place them side by side. We tend to rotate ourselves mentally about a vertical *axis, and that gives the diagram below,*



where R and L have swapped, but that's really an arbitrary choice.
If we rotated about a horizontal *axis, we would get the opposite result (see left) where the head and feet have swapped places.*

2.13.7 Rotations
Order of rotational symmetry is the number of times the shape will fit onto itself as it rotates through 360°.
Order 1 means no rotational symmetry.
A circle has infinite rotational symmetry, because you can stop it at any angle and it fits exactly onto itself.

To explain order of rotation you can use interlocking cubes which have holes in the middle. A pencil will fit through horizontally so that you can rotate the object about that axis.

Order is always a positive integer. (Actually, in quantum mechanics some particles – for example, an electron – have so-called "spin" of $\frac{1}{2}$, and this means they have to "rotate twice", 720°, to get back to where they started!)

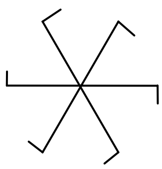
To rotate a shape without using tracing paper it often helps to join the centre of rotation to one of the vertices and rotate this line. You can do this for each vertex if necessary.

If using tracing paper, "centre of rotation" can be seen as the point on the tracing paper where you put your pencil – the point that doesn't move – everything else revolves around it.

Draw me a shape with order 6 rotational symmetry, but no reflection symmetry.

You may want to avoid unintentionally drawing Swastikas.

Lots of possibilities – you can lose the mirror symmetry by adding tails or flags to something with 6-fold symmetry.
e.g.,



What stays the same and what changes in a rotation?

Same: size, shape, lengths of sides, angles, area;
Different: orientation, position.

2.13.8 **NEED** a set of circuit symbols from a Physics/Electronics book.
What kind of symmetry do the symbols have?

Line symmetry may be parallel or perpendicular to the wire direction.

Answers: among the most common symbols, you find these symmetries:

- *no symmetry: variable resistor, variable capacitor, transistor, switch;*
- *line symmetry only: cell, ammeter, voltmeter, earth, diode, inductor, lamp (modern symbol);*
- *rotational symmetry (> order 1) only: source of alternating current;*
- *line symmetry and rotational symmetry (> order 1): connecting wire, lamp (old-fashioned cross symbol), resistor, transformer, fuse, capacitor.*

You can do the same with hazard warning symbols (Science department) (ignore the writing underneath the symbol).

"Toxic" has line symmetry, and "harmful" and "radioactive" have both line and rotational. "Oxidising" almost *has line symmetry but not quite because of the "flames"!*

Highway code road signs are another possibility, but most have no symmetry. Generally you should ignore the writing underneath and possibly ignore the shape of the sign itself (triangle, circle, etc.) as well.

- *line symmetry only: crossroads, dual carriageway ends, chevrons, road narrows on both sides, uneven road, traffic signals, hump bridge, level crossing with barrier, general danger, tunnel, low-flying air-craft, road humps;*
- *line symmetry and rotational symmetry: general warning, roundabout (line symmetry only approximate here).*

Signs from music notation.

- *line symmetry only: accents, ties, pause, crescendo, diminuendo, up/down bow (string players), alto/tenor clef;*
- *rotational symmetry only: sharp sign, natural sign, turn, mordent;*
- *line symmetry and rotational symmetry: breve, semibreve (and their rests), 5-line stave, bar line, double bar line, repeat marks, staccato dots, double-sharp sign.*

2.13.9 **NEED** crosswords from newspapers (collect for homework). Sort them according to their symmetry.

You could make a display out of this.

2.13.10 **NEED** pencil crayons (or just pencil), sheets.
Colour And Symmetry.
Colour in the shapes to give them rotational symmetry of

1. order 1
2. order 2
3. order 3

Answers:
Colouring the hexagon in the centre never makes any difference so long as it's all the same colour.

See sheet for answers.

What other orders of rotational symmetry are possible?

None.

What is the minimum number of different colours you need to use?

One (and white).

Do any of the finished shapes have any line symmetry?

No.

If using different colours, be careful not to embarrass anyone who is colour-blind.

2.13.11 Translations.
Pupils need to be clear that the vector defines the movement of each point to its image point; this isn't necessarily the same as the "gap" between the object and image shapes.

Translation vectors are not that difficult and are a less cumbersome way of describing translations than using words.
They are best defined as

$$\begin{pmatrix}\textbf{distance to the right}\\ \textbf{distance up}\end{pmatrix}.$$

(Notice that this is "upside down" compared with the way gradient is defined.)

What stays the same and what changes in a translation?

Same: size, shape, lengths of sides, area, angles, orientation;
Different: position.

2.13.12 Combined Transformations.

1. I'm thinking of a point. If I translate it by $\begin{pmatrix}4\\0\end{pmatrix}$, I get to the same point as if I reflect the point I'm thinking of in the y-axis. Where is the point? (There is more than one possibility.)

2. I'm thinking of a point. If I translate the point by $\begin{pmatrix}3\\-3\end{pmatrix}$, I get to the same point as if I reflect the point I'm thinking of in the lines $y = x$. Where could the point be?

3. I'm thinking of a point. If I translate the point by $\begin{pmatrix}-2\\-6\end{pmatrix}$, that's equivalent to a rotation of it by 90° clockwise about the origin. Where could the point be this time?

4. This time if I translate my point by $\begin{pmatrix}4\\2\end{pmatrix}$, that's equivalent to rotating it by 90° clockwise about the origin. Where is this point?

Answers:

1. *(–2, anything); i.e., any point on the line* $x = -2$;
2. *either (1,4) or (–4,–1);*
3. *(4,2)*
4. *(–3,1)*

Solve these by doing rough sketches.

In general, for questions 3 and 4, if a translation $\begin{pmatrix}a\\b\end{pmatrix}$ *is equivalent to a rotation 90° clockwise about the origin, then the co-ordinates of the point have to be* $(-\frac{1}{2}(a+b), \frac{1}{2}(a-b))$.

2.13.13 Enlargement.
What does "enlargement" mean?
Draw a 3 × 2 rectangle on the board.

"Gets bigger" – so draw a 10 × 2 rectangle;
"Gets bigger both ways" – so draw a 10 × 10;
"Gets bigger both ways by the same amount" – so draw a 4 × 3 rectangle, etc. (be awkward!).

Why are none of these proper enlargements?

It has to get the same proportion *(fraction) bigger both ways.*

If the "scale factor" is different in different directions, you get a stretch. You wouldn't be happy with this if your photos got "enlarged" like this – it isn't a proper enlargement.

Proportional thinking is always hard.

What stays the same and what changes in an enlargement?

Same: shape, angles, orientation;
Different: size, position, lengths of sides, area.

A reduction *sometimes counts as a (fractional) enlargement in maths.*

The scale factor number-line may be helpful here (see beginning of this section).

2.13.14 Accurate Enlargements.
You don't always need to have a centre of enlargement to draw an accurate enlargement; e.g., you can measure the sides and angles, keep the angles the same and multiply the lengths of sides by the scale factor.

Initially it's useful to use photocopied sheets so that you can be sure the enlarged shape will fit on nicely (see sheet).

What difference does it make if we move the centre of enlargement?
Same image shape except in a different place. (Centre of enlargement can even be inside the shape or on one of the vertices.)

Four possible "kinds" of scale factor (SF):
1. SF >1; shape gets bigger;
2. 0 < SF < 1; shape gets smaller;
3. SF < –1; shape gets bigger and inverted;
4. –1 < SF < 0; shape gets smaller and inverted.

In fact there isn't always a centre of enlargement even when a shape has been enlarged properly, because the new shape could have a different orientation from the original shape.

Scale factor (SF) can be positive or negative.

Emphasise that we make every measurement from the centre of enlargement. (If you measure from the corners of the original shape instead you get a SF + 1 enlargement.)

Pupils should check their own drawings by measuring the sides in the new shape (they should be SF × the lengths of the corresponding sides in the old shape), the angles (should be the same) and checking that the orientation is the same.

See the SF number-line at the beginning of this section.
Probably best to do in this order.

2.13.15 Enlargement. "Aspect Ratios", TV/cinema.
A normal TV screen has an "aspect ratio" of 4:3 (its size is 4 along by 3 up).
If you display a widescreen movie (2.35:1) so that the whole screen is filled with picture, what % of the picture do you lose?

What about if you view the whole picture (so you don't miss anything) "letterbox" style. What % of the screen is wasted with "black bars"?

Which do you think is better?

What if you have a high-definition TV (16:9)?

Answers:
% viewed = $\frac{4\times1}{2.35\times3} = 57\%$*, so 43% is missing.*
(You see all of the vertical direction but lose the two ends in the horizontal direction.)

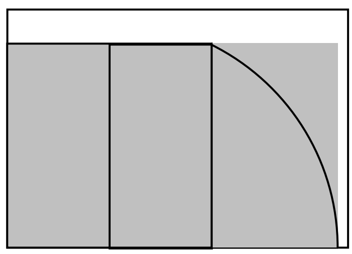
Same as before; 43% of the screen is black.

(Widescreen isn't always as "wide": 1.85:1 is common, as is 16:9, which is normal theatre screen dimensions.)

Film buffs tend to prefer to see everything the director intended, even if that means having a smaller picture.

This time you lose only 24% of the picture (or waste 24% of the screen).

2.13.16 **NEED c**ompasses, A4 plain paper.
Constructing a Golden Rectangle.
Take piece of A4 paper, landscape orientation, and draw a square 18 cm by 18 cm in the bottom left corner.
Split the square into two congruent rectangles with a vertical line.
Place the point of your compasses at the bottom of this line and stretch the pencil up to the top right corner of the square.
Draw an arc down from here until it reaches the bottom of the paper.
This point along the bottom side is the position of the bottom right end of the Golden Rectangle.
From here, draw a line 18 cm long vertically up the page. Then draw a line to meet the left side of the paper.



The shaded rectangle is called the "Golden Rectangle". Its sides are in the ratio $1:\phi$ *where* ϕ *= 1.61803… (*$\frac{1+\sqrt{5}}{2}$*, see below).*
(Pythagoras' Theorem gives the radius of the arc as $9\sqrt{5}$*, so the bottom length is* $9(1+\sqrt{5})$*.)*

2.13.17 **NEED** A4 white paper. Golden Ratio. Suitable homework. Draw 8 different-shaped rectangles on a blank piece of A4 paper (or cut out 8 different rectangles). Ask people to say which one or two look the "nicest" – "most pleasing to the eye".

Answer: People tend to choose the ones nearest to 3:2 or thereabouts. Some say that the rectangle most pleasing to the eye is the Golden Rectangle with sides in the Golden Ratio ($1:\phi$, see section 2.13.16). Renaissance artists may have used this to construct paintings.

2.13.18 **NEED** scrap paper, scissors.
I take a rectangle and fold over the shorter end so that it lies along the longer side. In this way I can mark off a square from the end. I cut off the square and the rectangle I'm left with, although it's obviously smaller, is the same shape (same dimensions) as the rectangle I began with. Can you find a rectangle that will do that. Will a 2:1 rectangle work?

(See task involving A-size paper in section 1.10.6.)

(See section 1.19.10 for a task involving the Fibonacci series.)

Answer:
2:1 will give 2 squares, so certainly not. If the sides are in the ratio $x:1$ ($x>1$), then algebraically $\frac{x-1}{1}=\frac{1}{x}$, so $x(x-1)-1=0$ or $x^2-x-1=0$, and the solutions are

$$x=\frac{1\pm\sqrt{1--4}}{2}, \text{ or } \frac{1+\sqrt{5}}{2},$$

since x must be positive.
So $x=1.61803...$
This is the "divine proportion" or "Golden Ratio". As indicated above, it is squared by adding 1 ($x^2=x+1$).

The ratio of a term in the Fibonacci sequence (1170-1250) to the previous term gets closer to the Golden Ratio as you go to higher and higher terms.

2.13.18 **NEED** A3 piece of paper showing a large footprint or "pawprint". What can you say about the size of the animal that could have produced this?! (Imagine we discovered it outside school in the morning.)

Pupils can take measurements from it and try to predict things like height, mass, length of stride, the tallest wall it could climb over, how much food it might eat per day, etc.

2.13.19 On squared board or 1 cm × 1 cm squared acetate, draw two separate 2 × 2 squares.

Suitable for reviewing the transformations topic.

Lots of possibilities: translation, rotation, reflection followed by translation, reflection followed by a different reflection, etc.

The white square has become the grey square. What's happened to it, apart from the change in colour?

Pupils can try to describe the transformations as precisely as possible.

Label the white square ABCD. How would you have to label the grey square (where would you put A', B', C' and D' to make it each of the transformations pupils have suggested?

The white square is the object; the grey square is the image.

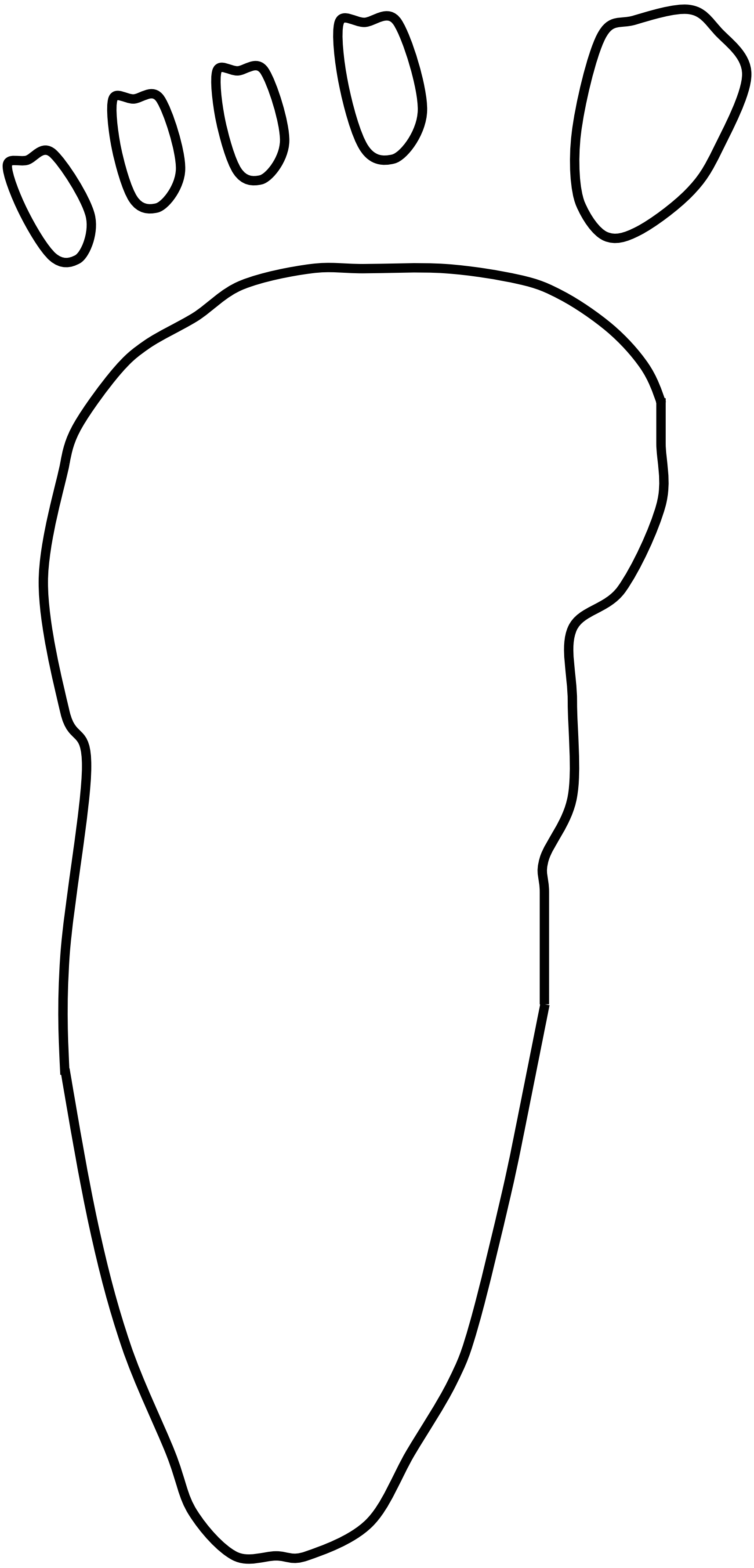
2.13.20 You could begin a lesson by writing something like this on the board:

What do you think this lesson
is going to be about?

Be prepared to help those who may find this very hard.

You can give instructions in this way to pupils at the start of a lesson on reflections; e.g.,

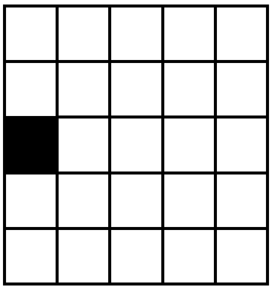
Don't say anything.
If you can understand this, then turn
to the back of your exercise book
and write your first name and last
name in reflection writing.



Symmetrical Squares

In each drawing, shade in *exactly 3 more* squares so that the whole drawing ends up with *exactly 2 lines* of symmetry.

1



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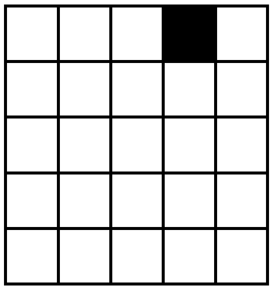

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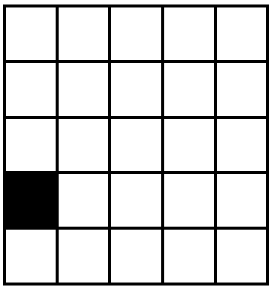

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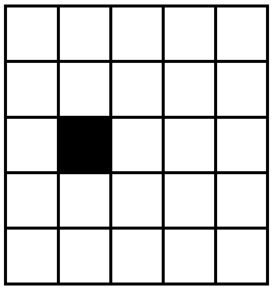

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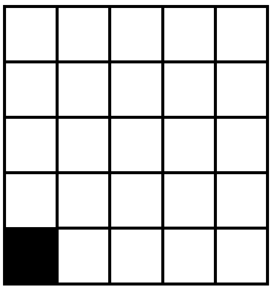

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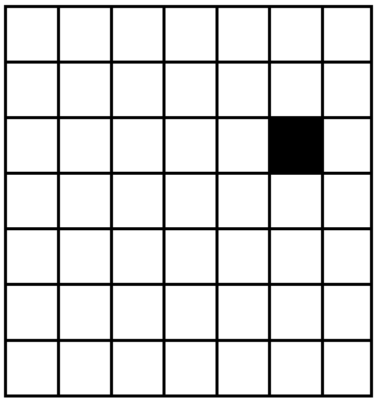

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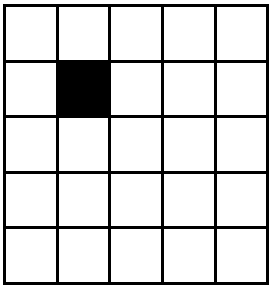

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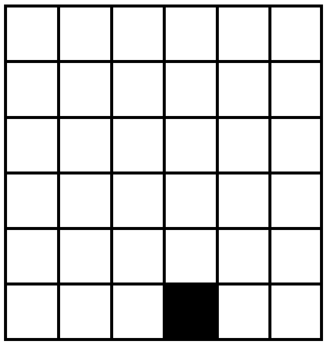
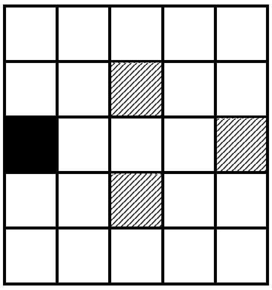


Symmetrical Squares ANSWERS

In each drawing, shade in *exactly 3 more* squares so that the whole drawing ends up with *exactly 2 lines* of symmetry.

There are many possibilities; only one answer is shown for each question.
All of the shaded squares would have to be the same colour.

1



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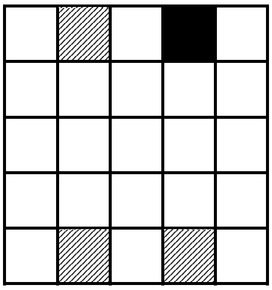

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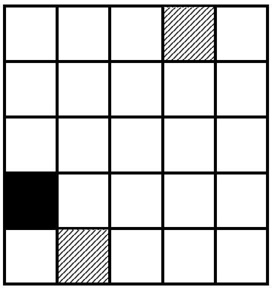

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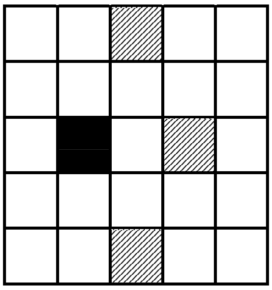

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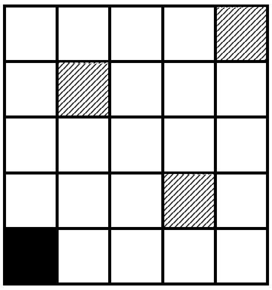

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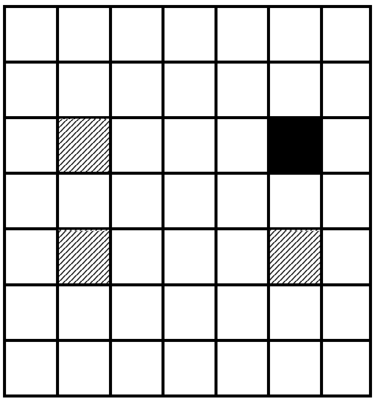

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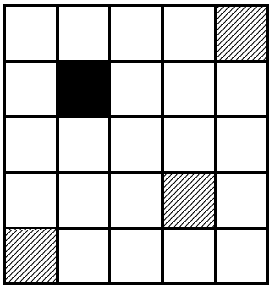

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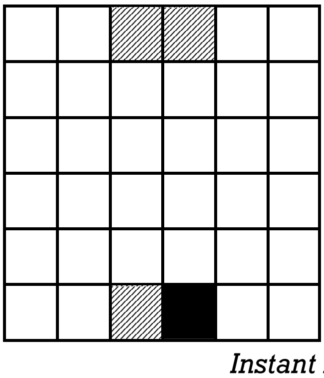
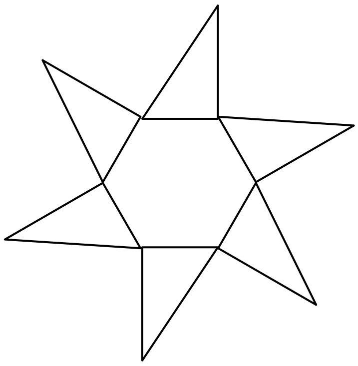
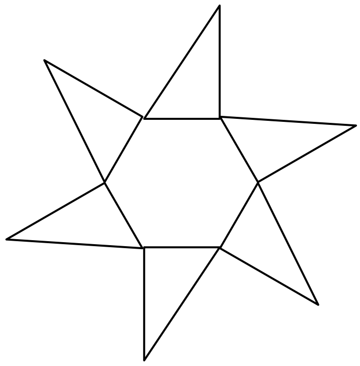


Colour And Symmetry

Colour these shapes so that they have different orders of rotational symmetry.



Order ______



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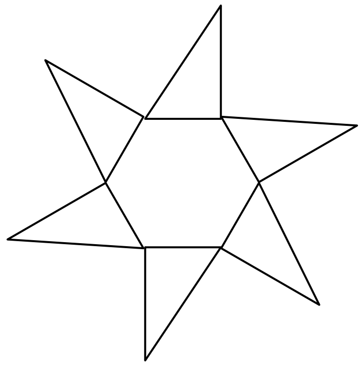

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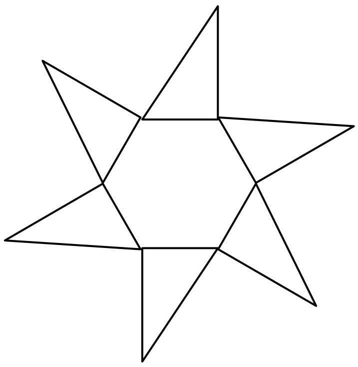

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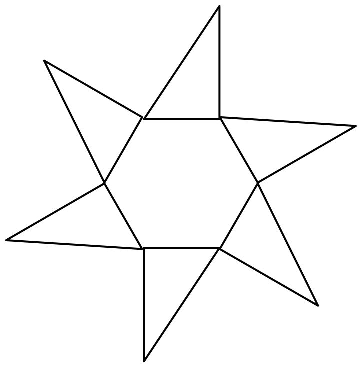

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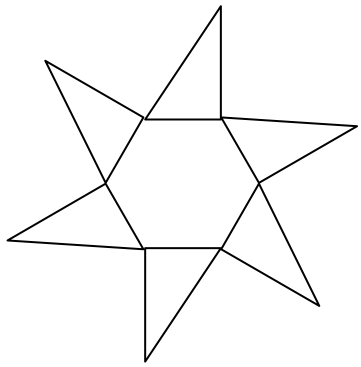

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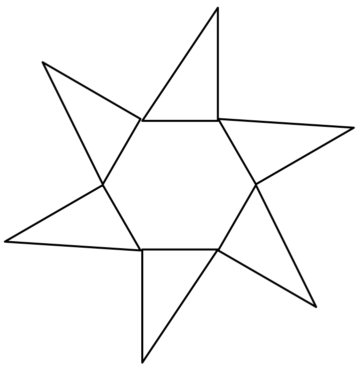

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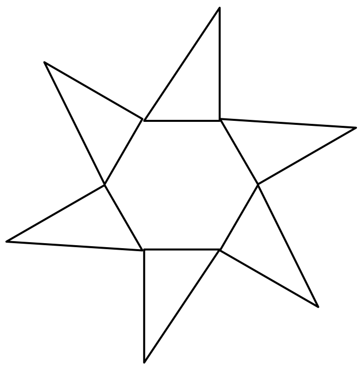

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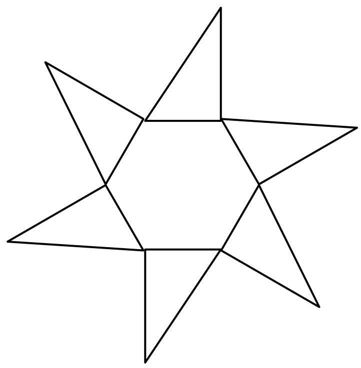

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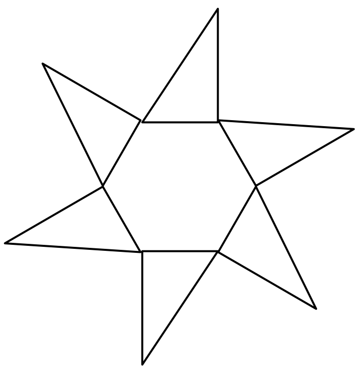

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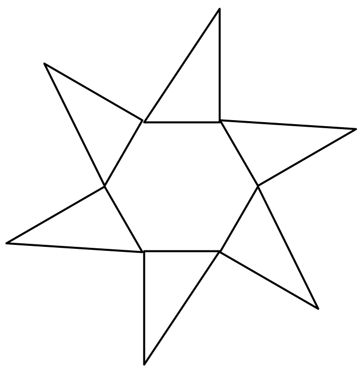

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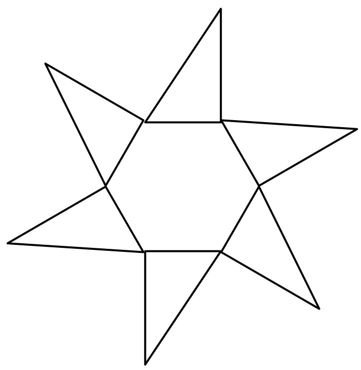
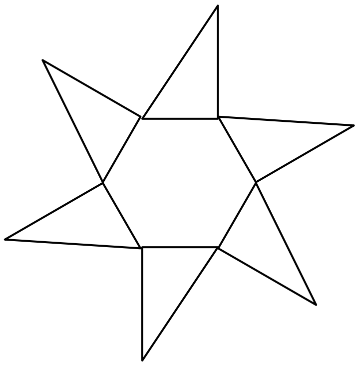


Order ______

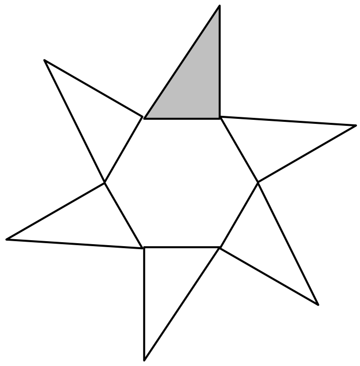
Colour And Symmetry ANSWERS

Colour these shapes so that they have different orders of rotational symmetry.

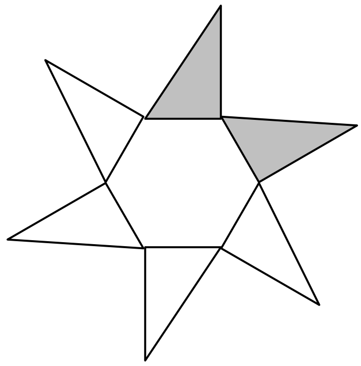
These are the possibilities using just one colour (and white).
In each case, the shaded and white areas could be swapped (making a "negative"), usually giving a different answer with the same order of rotational symmetry.



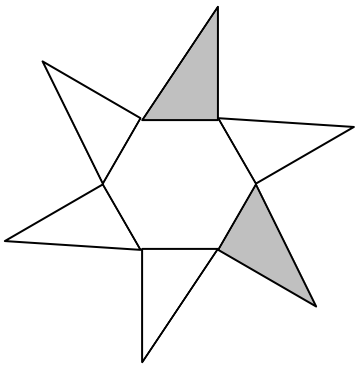
Order 6



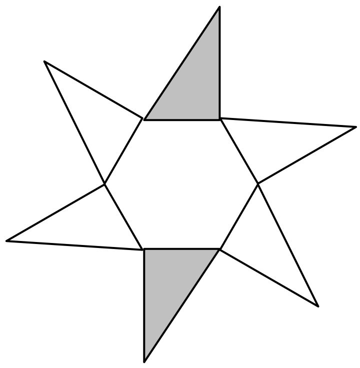
Order 1



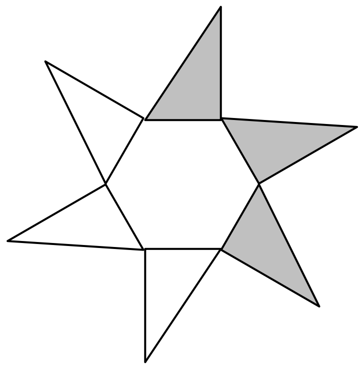
Order 1 ("ortho")



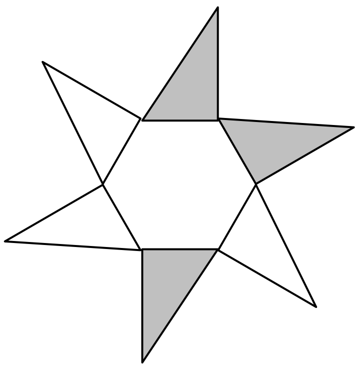
Order 1 ("meta")



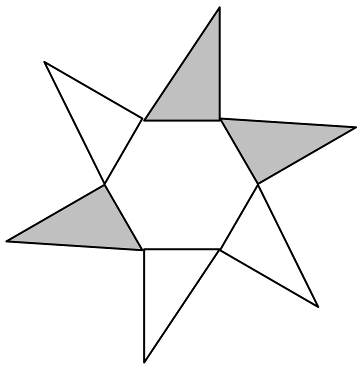
Order 2 ("para")



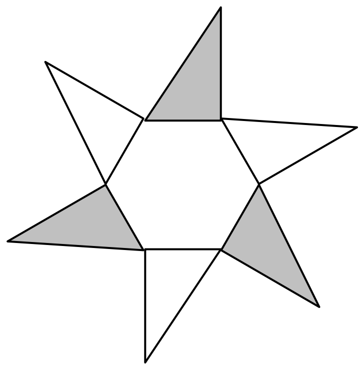
Order 1



Order 1



Order 1



Order 3

These two (above) have "negatives" which are the same as themselves but they are not the same as each other.

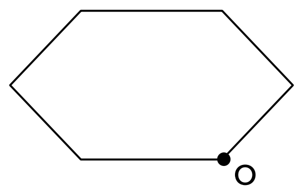
The names "ortho", "meta" and "para" refer to substitution patterns in derivatives of the chemical molecule benzene, which has a planar hexagonal shape and 6-fold symmetry.

Drawing Accurate Enlargements

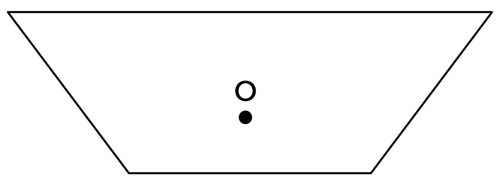
Enlarge these shapes as accurately as you can, using O as the centre of enlargement.
None of the enlargements should go off the page.

1 Scale factor 2

2 Scale factor 3



3 Scale factor 2



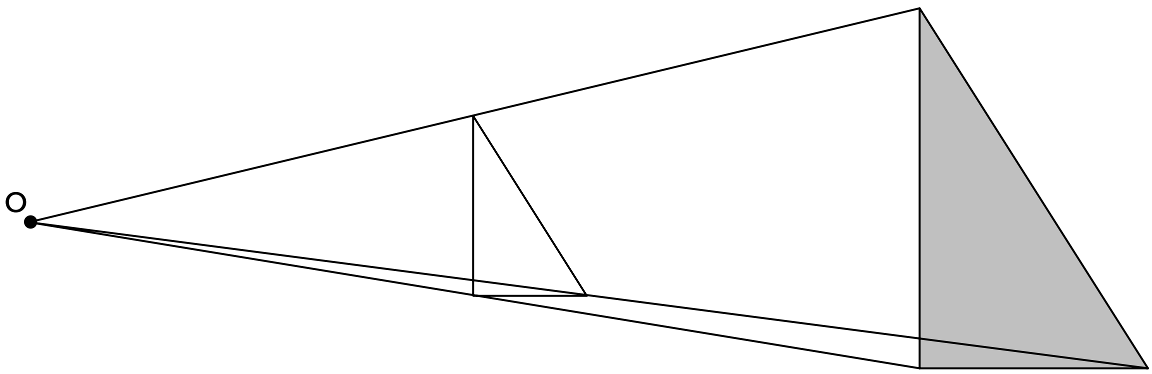
4 Scale factor 1.5

Drawing Accurate Enlargements **ANSWERS**

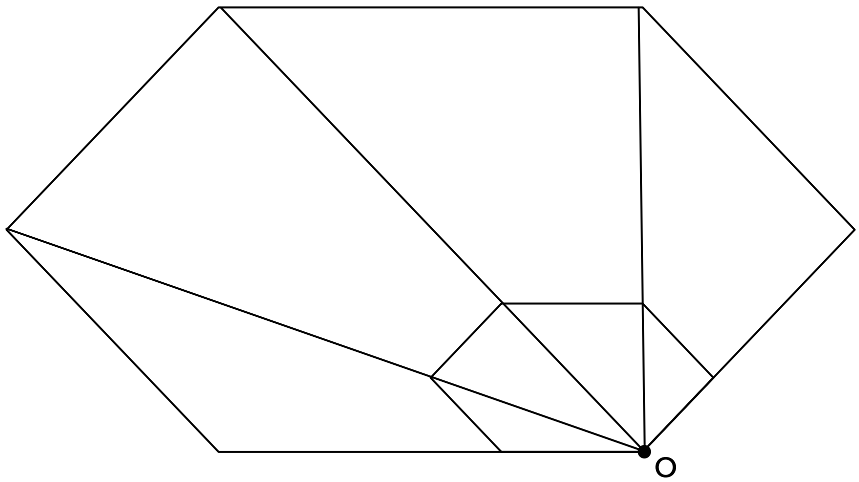
Enlarge these shapes as accurately as you can, using O as the centre of enlargement.
None of the enlargements should go off the page.

(You could photocopy this sheet onto an acetate and place it over the pupils' work to mark.)

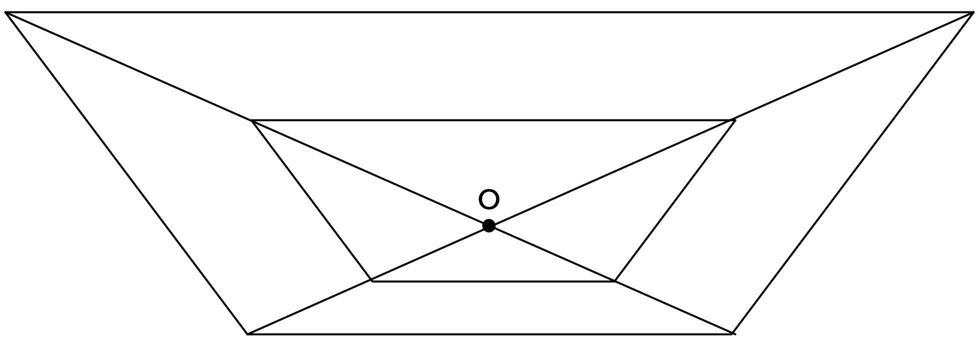
1 Scale factor 2



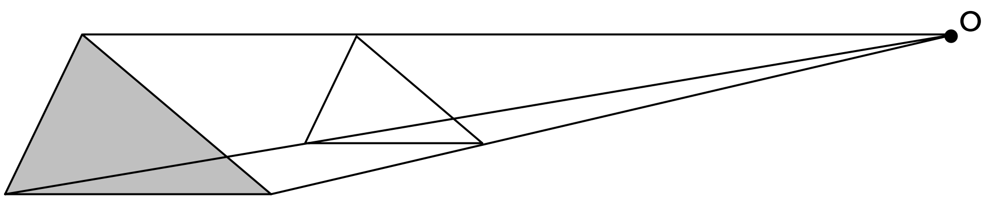
2 Scale factor 3



3 Scale factor 2



4 Scale factor 1.5



2.14 Tessellations

- A misconception here is that the interior angles in shapes that tessellate have something to do with the *factors* of 360. That cannot be right, because 360 is an arbitrary number that we choose to divide a whole turn into. We could have used 100, or 2π (could mention radians here), or 359, which is prime. See sheet for an explanation of why only some polygons tessellate.

2.14.1 Polyominoes.
Obviously dominoes will tessellate because they are just rectangles.
What about both of the triominoes?
What about the tetrominoes?
And so on …

Answers:
All triominoes, tetrominoes, pentominoes and hexominoes tessellate.
From heptominoes onwards they don't all tessellate.

See section 2.2.9 for how many of each of the polyominoes there are.

2.14.2 What kinds of triangles tessellate?

Answer: all triangles tessellate, because any two congruent triangles will make a parallelogram if you put a pair of corresponding sides together, and parallelograms tessellate.

2.14.3 What kinds of quadrilaterals tessellate?

Answer: again all will, even concave ones.

2.14.4 Design a tessellating shape. You can "force" it to tessellate by starting with something that certainly tessellates (e.g., a parallelogram) and doing opposite things to opposite sides (e.g., cut out a triangle from one side and add it on to the opposite parallel side).

This can make good display work.

Christmas trees are possible.

2.14.5 **NEED** cardboard or plastic polygons.
(You could use the polygon shapes from section 2.1.)
Draw round them and see which ones you can get to tessellate.
To start with, try only one type of regular polygon in each pattern. You should find 3 "regular tessellations". (See sheet.)

It really is worth using cardboard (the thicker the better) and not paper to make templates, because they are much easier to draw round. You can get a lot out of one A4 sheet of card.

2.14.6 **NEED** Escher (1898-1972) drawings (see books).

Remarkable examples of intricate tessellations.

2.14.7 Where have you seen beautiful tessellations?

Answers (suggestions):
Islamic art, mosaics (are there some in school or could people bring in photos?).

2.14.8 Where are tessellations not just pretty but useful?

Answers (suggestions):
1. *bee-hive: very sensible structure (rigid);*
2. *some molecular structures (e.g., graphite is made up of sheets of tessellating hexagons of carbon atoms);*

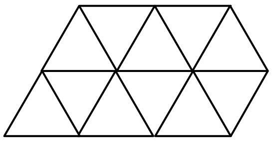
Answers (continued):
3. *squared/isometric paper;*
4. *rigid bridge structures (equilateral triangles);*
5. *kitchen tiles (no gaps is important, because water would get through).*
6. *paving slabs, brick walls.*

Tessellations

A tessellation is a pattern of shapes which cover all of the surface with no gaps and no overlapping.

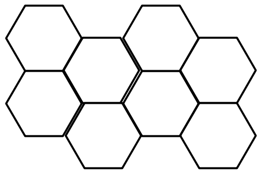
There are 3 Regular Tessellations – all the shapes are the same regular polygon and all the vertices are the same.

Equilateral Triangles



Squares

Hexagons

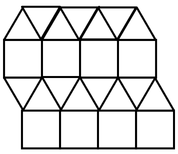


In a regular polygon with n sides, each interior angle is $\frac{180(n-2)}{n}$.

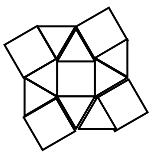
If m (a positive integer) of them meet at a point, then $\frac{180(n-2)m}{n} = 360$, and this simplifies to $\frac{m(n-2)}{n} = 2$, or $m = \frac{2n}{n-2}$. So if $\frac{2n}{n-2}$ is an integer, the regular polygon will tessellate.
This happens only when n = 3, 4 or 6 (equilateral triangles, squares and hexagons).

There are 8 Semiregular Tessellations – all the shapes are regular polygons, but they're not all the *same* regular polygon. All the vertices are still the same.

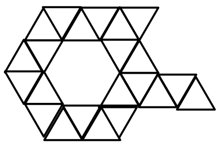
3,3,3,4,4



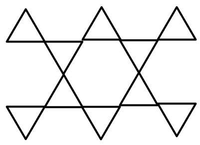
3,3,4,3,4



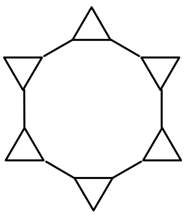
3,3,3,3,6



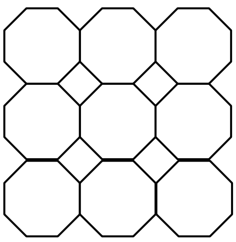
3,6,3,6



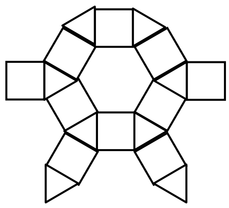
3,12,12



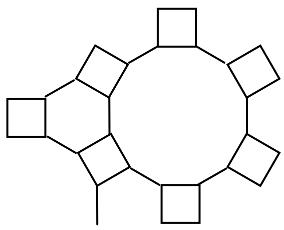
4,8,8



3,4,6,4



4,6,12



Every vertex has the same arrangement of regular polygons around it.
Going clockwise or anticlockwise around a vertex, the number of sides on each of the polygons present make the sequences of numbers above (e.g., 3,3,3,4,4 means that at each vertex you have triangle-triangle-triangle-square-square).

2.15 Dimensions and Units

- Although the practical everyday relevance is clear, this can be a dull topic unless there is some purpose to converting quantities from one unit to another. This topic works best by combining with others; e.g., standard form, volume/area, estimation, etc. There are some suggested problems below.
- It's hard to say exactly what "dimensions" are. You could ask pupils if they know what the "d" stands for in "3d" and then see what they think there are "3" of. The answer is something like "mutually perpendicular directions". Mathematicians often talk about 4 or more dimensions. In Maths, extra dimensions often don't make things that much harder to calculate, but it gets harder/impossible to visualise!

2.15.1 Conversion graphs.
Find out currency conversion rates from newspapers or the internet.

Bringing along foreign coins adds interest. See if pupils can identify the country and estimate how much the coin is worth in our money.

Pupils can draw, for example, value in French Francs against value in British Pounds on one graph, and German Marks against British Pounds on another. Pupils can then convert Francs to Marks using one graph after another (pick several values) – the resulting graph of Marks against Francs should also be a straight line through the origin.

It's much easier to be a millionaire in some countries than in others!

Could discuss stock-markets, inflation, etc.

Are conversion graphs always straight lines?

(Actually, time in seconds = $\sqrt{\frac{d}{490}}$*, where* d *= distance fallen in cm.)*

Not necessarily; e.g., dropping a ruler between someone's fingers to measure their reaction time – the graph to convert cm to seconds is a curve.

(Each 1 cm fallen counts for less as time goes on, because the ruler is speeding up.)

Do they always go through the origin?

Again, not necessarily; e.g., °C to °F.

2.15.2 Which is bigger, an imperial ton or a metric tonne? Are they different in the UK and the US? (Could find out for homework.)

Answer: Imperial is spelt "ton"; metric is spelt "tonne".
A UK ton is 2240 lb (a so-called "long ton" or "gross ton"), whereas a US ton (a "short ton" or "net ton") is only 2000 lb.

So the order is
UK ton > metric tonne > US ton.

Since a metric tonne is 1000 kg (anywhere!), and there are 2.205 lb in a kg, a metric tonne is 2205 lb, so this is in between (see left).

What about gallons?
Similarly, US pints are less than UK ones, but US fluid ounces are more*, since in the US there are 16 fluid ounces in a pint, whereas in the UK there are 20!*

A UK gallon is 4.55 litres, whereas a US gallon is only 3.79 litres.

2.15.3 Estimate the total mass of everyone in the room?

Answer: Assume an average pupil weighs 50 kg. A class of 30 would weigh 30 × 50 kg = 1500 kg or 1.5 tonne.

What about the total mass of everyone in school assembly?

Depends on the size of the school, obviously. (Be cautious if anyone might be sensitive about this task.)

2.15.4 Find out how high up aeroplanes typically fly.

How high are the tallest buildings?

How high up are satellites?

How far away is the moon/the sun?

Answers: (Note that 5280 ft = 1 mile.)

- ***aeroplanes**: e.g., 30 000 ft = 6 miles (approx) (The SR71 spy-plane flew at an altitude of 16 miles, but the pilots had to wear space-suits!);*
- ***tallest buildings**: (lots of debate over exactly what counts) around 500 m or nearly 2000 ft;*
- ***satellites**: anywhere from 100's of miles to tens of thousands of miles; e.g., geostationary satellites are at 22 223 miles (the further out they are the longer they last because there's less material for them to bump into);*
- ***moon** (a natural satellite): 240 000 miles;*
- ***sun** (a star): 93 000 000 miles.*

Can you draw a scale diagram to illustrate? (possible homework)

It's impossible to draw them all on a linear scale.

Find out how astronomers measure distances?

*The mean distance from the earth to the sun is called an **"astronomical unit" (AU)**, 1.5×10^{11} m, or 9.3×10^{7} miles. For example, astronomers might say that the distance of mercury from the sun is 0.39 AU, whereas for Pluto it is 40 AU.*
***"Light years" (ly)** are another way of measuring* distance *(not time); a light year is the distance light travels in a vacuum in one year and is 10^{16} m approx.*
*Astronomers also use **"parsecs" (pc)**, and 1 parsec = 3.26 ly = 3×10^{16} m.*

What about leagues and fathoms?

They're used in sea-travel.
1 fathom = 6 feet;
1 league = 3 miles
(1 nautical league = 3 nautical miles;
a nautical mile = 1.15 land miles.)

2.15.5 Estimate the number of tubes of toothpaste used per year in the UK.

What assumptions do you have to make?

Answer: Assume that there are 60 million people in the UK and that everyone brushes their teeth on average once a day (some more, some less). Assume all tubes hold 75 g toothpaste and that everyone uses 1 g for each brushing.
Therefore, for 365 days (leap years make no significant difference) we'll use $365 \times 1 \times 60 \times 10^{6}$ g = 2×10^{10} g, which corresponds to $2 \times 10^{10} \div 75$ tubes = 3×10^{9}, 3 billion tubes per year (approx).

2.15.6 How many pencils would it take to stretch across a football pitch from one goal to the other?
How many to stretch to the top of the Eiffel Tower?
How many to stretch a mile?
How many to go all the way round the world at the equator?
How many to go to the moon and back?

Answers:
Take an average pencil as 15 cm long.
***Football pitch** = 100 m long, so about 700.*
***Eiffel Tower** = 324 m high, so about 2000.*
***A mile** = 1600 m, so about 11 000 pencils.*
***Equator** = $2\pi r$ where r = radius of the earth = 6.4×10^{6} m, so equator = 4×10^{7} m so about 3×10^{8} pencils (300 million).*
*Average **distance to the moon** = 4×10^{8} m, so twice this is 8×10^{8} m, so about 5×10^{9} pencils (5 billion).*

2.15.7 Dimensions.
Tell me a kind of shape and how to work out its area. We'll write it as a formula.
e.g., square: l^2 ; triangle: $\frac{1}{2}bh$; etc.
Now tell me some solids and how to work out their volumes.
What do you notice?

You can't do this topic until pupils are familiar with finding areas and volumes of a number of different shapes/solids.

Record in two columns on the board: "area formulas"; "volume formulas".

Area is always found by multiplying two lengths together (possibly also multiplying by a fixed number); Volume is always a length multiplied by a length multiplied by length, or an area multiplied by a length (and possibly multiplied by a constant).

If we write this as L^2 and L^3, then this just means "some length" squared/cubed.

The formula for the area of an ellipse is one of these. Which one?
πabc ; πab ; $\pi a^2 b$; πab^2 ; $\pi(a+b)$

πab, since this is the only formula with L^2 dimensions. (a is half the length of the major axis (longest diameter) and b is half the length of the minor axis (shortest diameter).)

It's worth reinforcing the point that $A \Rightarrow B$ does not mean that $B \Rightarrow A$. Right formula $\Rightarrow$ right dimensions, but right dimensions $\not\Rightarrow$ right formula.
$A \Rightarrow B$ does mean that $B' \Rightarrow A'$ (where B' means "not-B"). So dimensions wrong $\Rightarrow$ formula definitely wrong.

Using dimensions never helps us to get the constant right, for example.
So dimensions would never tell us to put in the π in the formula for the area of the ellipse.

In Mechanics, you also use M and T for mass and time. Other areas of Science require temperature θ, current A and even luminous intensity I. Most things can be made up from combinations of these, or else they're dimensionless (e.g., angles).

2.15.8 Check out the "dimensional soundness" of some Physics formulas; e.g.,

See Physics books for definitions of these quantities.

- Newton's 2nd Law: $F = ma$
  $[F] = \text{Newtons}$
  $[ma] = MLT^{-2}$
  so 1 Newton is defined as 1 kg m/s^2;
- constant acceleration formulas; e.g., $v = u + at$ and $v^2 = u^2 + 2as$;
- work done and energy formulas; e.g., $W = Fs$ and $E = \frac{1}{2}mv^2$;
- the time period of a pendulum: $T = 2\pi\sqrt{\frac{l}{g}}$;
- wave motion: $v = f\lambda$;
- lenses: $\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$;
- electricity; e.g., $V = IR$, $P = VI$, $Q = CV$;
- magnetism; e.g., $T = BANI\cos\alpha$;
- fields; e.g., $F = -\frac{Gm_1m_2}{r^2}$, $F = \frac{Q_1Q_2}{4\pi\varepsilon_0 r^2}$;
- pressure; e.g., $pV = nRT$, $p = \rho hg$;
- radioactivity; e.g., $N = N_0e^{-\lambda t}$;

and many others.

- *You can think of this formula as defining a Newton as the force necessary to accelerate a 1 kg mass by 1 m/s^2;*
- $[at] = LT^{-2}T = LT^{-1}$;
- $[Fs] = MLT^{-2}L = ML^2T^{-2}$ *= Joule;*
- $\left[\frac{1}{2}mv^2\right] = M(LT^{-1})^2 = ML^2T^{-2}$;
- $\left[2\pi\sqrt{\frac{l}{g}}\right] = \left(\frac{L}{LT^{-2}}\right)^{\frac{1}{2}} = \left(\frac{1}{T^{-2}}\right)^{\frac{1}{2}} = T$;
- $[f\lambda] = T^{-1}L = LT^{-1} = [v]$;
- $[u] = [v] = [f]$;
- $[V] = \frac{ML^2T^{-2}}{AT} = A \times ML^2T^{-3}A^{-2}$ *, etc.;*

Dimensions can help with remembering the units of constants such as
$G = 6.67 \times 10^{-11}$ N m^2 kg^{-2},
$\varepsilon_0 = 8.85 \times 10^{-12}$ F m^{-1} *and* $h = 6.63 \times 10^{-34}$ J s.

Metric and Imperial Measurements

	Length	Volume	Mass	
METRIC	**mm** (÷10 →, ← ×10) **cm** (÷100 →, ← ×100) **m** (÷1000 →, ← ×1000) **km**	**cm^3** = **ml** (÷1000 →, ← ×1000) **litre** (÷1000 →, ← ×1000) **m^3**	**mg** (÷1000 →, ← ×1000) **g** (÷1000 →, ← ×1000) **kg** (÷1000 →, ← ×1000) **tonne**	**METRIC**
CONVERSION	**cm** (÷2.5 →, ← ×2.5) **inch**; **km** (÷1.6 →, ← ×1.6) **mile**	**pint** (÷1.7 →, ← ×1.7) **litre** (÷4.5 →, ← ×4.5) **gallon**	**g** (÷28 →, ← ×28) **oz**; **lb** (÷2.2 →, ← ×2.2) **kg**	**CONVERSION**
IMPERIAL	**inch** (÷12 →, ← ×12) **foot** (÷3 →, ← ×3) **yard** (÷1760 →, ← ×1760) **mile**	**floz** (÷20 →, ← ×20) **pint** (÷8 →, ← ×8) **gallon**	**oz** (÷16 →, ← ×16) **lb** (÷14 →, ← ×14) **stone** (÷160 →, ← ×160) **ton**	**IMPERIAL**

2.16 Compound Measures and Rates of Change

- To use time in calculations (e.g., working out speed) pupils need to convert hours and minutes either to decimal hours or to minutes (see the first task below).
- Don't assume all pupils will be confident reading an analogue clock.
- See section 1.25 for further ideas.

2.16.1 Decimal Time.
If I went on a journey and said it took me 3.25 hours, why might that be confusing?
How long do I really mean?

Answer: Do I mean 3 hours and 25 minutes or do I mean $3\frac{1}{4}$ hours?
3.25 h = 3h15min

How long is 3.7 hours literally?

Less than 3h45min (3.75 h, $3\frac{3}{4}$ h).
More than 3h30min (3.5 h, $3\frac{1}{2}$ h).

Decimal time × 60 = time in minutes.

Could say that 0.1 h = 6 min, so 0.7 h = 42 min, so the time is 3h42min.

2.16.2 **NEED** local train/bus timetables (companies will sometimes give you as many as you want at no cost, especially if they're nearly out of date).

You may need to explain how the timetables work; i.e., different sides for different directions; "slow" and "fast" trains; different services Saturday/Sunday, etc.

Oral and mental work based on times.
e.g., "I want to get to London by 6 pm. Which train should I catch and how long will the journey take?"

Pupils need to apply commonsense bearing in mind that services may be delayed or cancelled.

2.16.3 What does it mean if an aeroplane travels at "mach 2.5"?

Answer:
The "mach" number (named after Ernst Mach, 1838-1916) is the number of times the speed of sound that the aeroplane is travelling.
Mach >1 means "supersonic".

The speed of sound in air is 330 m/s = 760 mph at sea level, but it drops considerably with altitude (e.g., it's only 590 mph at 30 000 ft) because of the decrease in density.

(You have to say the speed of sound in air *because sound waves need something to go through – the speed of sound in a vacuum is zero.)*

2.16.4 Do you think there's a limit to how fast any object can go?

Of course ordinary objects (e.g., an aeroplane) would fall to bits if we tried to make them go too fast, but Einstein's theory is more fundamental than that.

Answer:
According to Einstein's (1879-1955) relativity theory, no object can go faster than the speed of light in a vacuum (c).
$c = 3 \times 10^8$ m/s or 186 000 miles/s.
Sometimes other speeds are given relative to c; e.g., speed of electrons in a particle accelerator could be $0.9c$.

2.16.5 When is speed measured in knots?

Answer:
It's a unit of speed often used for aircraft and boats; 1 knot = 1 nautical mile per hour = 1.15 land miles per hour.

2.16.6 "Around the World in 80 Days", Jules Verne.
What was Phileas Fogg's average speed?

Answer: $\frac{4\times10^4}{80\times24} = 21$ *kph (approx).*

2.16.7 Would you say we're moving at the moment? The earth is rotating. Estimate how fast you think we're moving (mph).
What would you need to know to work out our speed?
Radius of earth = 6.4×10^6 m
(Hint: Imagine we're on the equator.)

Why doesn't it feel like it?

Of course, the earth is also orbiting the sun.

Answer:
On the equator, we move $2\pi r$ *metres every 24 hours, which is* $2\pi \times 6.4 \times 10^6 = 4 \times 10^7$ *m, corresponding to a speed of about 1700 kph or 1000 mph. Off the equator it's slower.*

The angular speed is very low (0.25° per min), so we don't notice our direction changing. We can't tell the high speed because the atmosphere, etc. moves with us (like being on a very smooth train at night).

2.16.8 It takes 8 hours to fly from London to New York, a distance of 3 500 miles. What is the average aeroplane speed?
Concorde gets there in about $3\frac{1}{2}$ hours. What is Concorde's average speed?

If Concorde could fly non-stop around the world, how long would it take?

Answers:

440 mph (sub-sonic)

1000 mph (supersonic; Concorde cruises at about Mach 2)

$\frac{4 \times 10^4 \div 1.6}{1000} = 25$ *hours (just over a day).*

2.16.9 I am standing on the platform at a railway station. An inter-city train speeds through the station and it takes 4 seconds to pass me. A few moments later, another train of the same length comes through going the other way. This second train takes 5 seconds to pass me. How long did it take them to pass each other?

Answer: If x *is the (unknown) length of the trains, then the speed of the first train is* $\frac{x}{4}$ *and the speed of the second is* $\frac{x}{5}$*. Their speed relative to each other will therefore be* $\frac{x}{4} + \frac{x}{5} = \frac{9x}{20}$*. When they pass each other there is a relative distance of* $2x$ *to cover, so the time taken will be* $2x \div \frac{9x}{20} = \frac{40}{9} = 4\frac{4}{9}$ *seconds.*

2.16.10 Alison and Beckie run a 100 m race. Alison wins by exactly 1 m. If they run again, but this time Alison starts 1 m behind the starting line, who will win this time?
Assume that they both run at steady speeds and perform just as well on the second race.

Answer: Alison again. When Alison runs her first 100 m, Beckie will have got to 99 m, so they'll be level. But then in the next 1 m Alison will overtake and win by 1 cm.

2.16.11 If sound travels at 330 m/s, make up an easy to remember rule (or check one you already know) to work out how far away lightning is when you see the flash and hear the thunder.

Answer: The speed is roughly 1 km every 3 seconds, so one possibility would be "count the seconds from the flash to the thunder – could say 'zero' on the flash – and divide by 3 to find out the distance away in km".

2.16.12 Density. Which weighs more, 1 kg of wood or 1 kg of steel?
What is different about 1 kg of wood and 1 kg of steel?

Work out the mass of a cuboid gold bar that is 18 cm by 9 cm by 4.5 cm.
The density of gold is 19.32 g/cm^3.
Do you think you could lift one?

Could work out how much it would be worth. Prices of gold vary minute by minute, but they're in the region of £7 000 per kg.

What would be the value of a silver bar the same size? (The density of silver is 10.49 g/cm^3; the cost is roughly £100 per kg.))

Answer: the same, of course!
The steel would take up much less space (volume) than the wood would.

Volume = 18 × 9 × 4.5 = 729 cm^3.
So mass = 729 × 19.32 = 14 kg (or about 30 lb).

Yes. About 14 bags of sugar, or half a sack of potatoes!

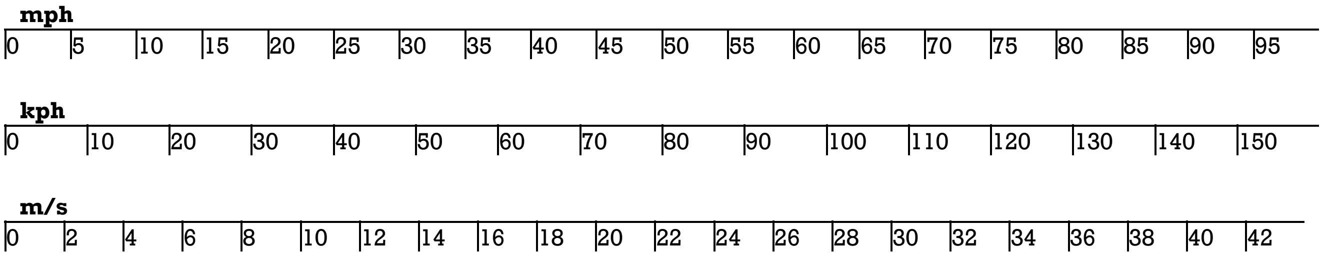
This would give a value of about £100 000.

You could work it out as above, or scale down.
cost = $100000 \times \frac{10.49}{19.32} \times \frac{100}{7000}$ *= £ 800 approx.*

2.16.13 How dense are we?!
Average human volume is about 70 litres (see section 2.10.14) and average human mass is around 70 kg, so average human density is about 1 kg/litre or 1 g/cm^3. This is the density of water, and that explains why we float, but only just.

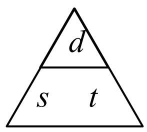
This is "average" human density in two senses. Not all human beings are identical, of course, but also the body is non-uniform. Bones are dense and sink, whereas lungs are relatively light. So this is average density over the whole body as well as the whole population.

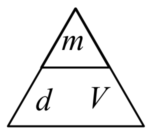
Speed Measured in Different Units



	÷1.15 →		×1.6 →		$\times\frac{5}{18}$ →	
knot		**mph**		**kph**		**m/s**
	← ×1.15		← ÷1.6		← $\div\frac{5}{18}$	

The triangle on the left gives the formulas for speed s, distance d and time t.
The one on the right gives the formulas for density d, mass m and volume V.





Densities of Common Materials

material	density (g/cm^3)	mass of 50 cm^3	mass of 35 cm^3	volume of 50 g	volume of 35 g
water	1.00	50.0	35.0	50.00	35.00
aluminium	2.70	135.0	94.5	18.52	12.96
zinc	7.13	356.5	249.6	7.01	4.91
iron	7.87	393.5	275.5	6.35	4.45
copper	8.96	448.0	313.6	5.58	3.91
silver	10.49	524.5	367.2	4.77	3.34
lead	11.36	568.0	397.6	4.40	3.08
mercury	13.55	677.5	474.3	3.69	2.58
gold	19.32	966.0	676.2	2.59	1.81

Key Stage 3 Strategy – Key Objectives Index

These are the key objectives from the *Key Stage 3 Strategy* (DfES, 2001) with references to sections of relevant material from all three volumes.

Year 7

Simplify fractions by cancelling all common factors; identify equivalent fractions.	1.6
Recognise the equivalence of percentages, fractions and decimals.	1.11
Extend mental methods of calculation to include decimals, fractions and percentages.	1.2-11, 3.6
Multiply and divide three-digit by two-digit integers; extend to multiplying and dividing decimals with one or two places by single-digit integers.	1.5, 3.6
Break a complex calculation into simpler steps, choosing and using appropriate and efficient operations and methods.	various
Check a result by considering whether it is of the right order of magnitude.	1.15, 2.15-16
Use letter symbols to represent unknown numbers or variables.	1.19-22, 1.26
Know and use the order of operations and understand that algebraic operations follow the same conventions and order as arithmetic operations.	1.12, 1.20
Plot the graphs of simple linear functions.	1.23
Identify parallel and perpendicular lines; know the sum of angles at a point, on a straight line and in a triangle.	2.4-5
Convert one metric unit to another (e.g., grams to kilograms); read and interpret scales on a range of measuring instruments.	2.15, 1.2
Compare two simple distributions using the range and one of the mode, median or mean.	3.3
Understand and use the probability scale from 0 to 1; find and justify probabilities based on equally likely outcomes in simple contexts.	3.5
Solve word problems and investigate in a range of contexts, explaining and justifying methods and conclusions.	various

Year 8

Add, subtract, multiply and divide integers.	1.3, 3.6
Use the equivalence of fractions, decimals and percentages to compare proportions; calculate percentages and find the outcome of a given percentage increase or decrease.	1.9-11
Divide a quantity into two or more parts in a given ratio; use the unitary method to solve simple word problems involving ratio and direct proportion.	1.10
Use standard column procedures for multiplication and division of integers and decimals, including by decimals such as 0.6 or 0.06; understand where to position the decimal point by considering equivalent calculations.	1.2-3, 1.5, 3.6
Simplify or transform linear expressions by collecting like terms; multiply a single term over a bracket.	1.20
Substitute integers into simple formulas.	1.20
Plot the graphs of linear functions, where y is given explicitly in terms of x; recognise that equations of the form $y = mx + c$ correspond to straight-line graphs.	1.23
Identify alternate and corresponding angles; understand a proof that the sum of the angles of a triangle is 180° and of a quadrilateral is 360°.	2.4
Enlarge 2-d shapes, given a centre of enlargement and a positive whole-number scale factor.	2.12-13
Use straight edge and compasses to do standard constructions.	2.8
Deduce and use formulas for the area of a triangle and parallelogram, and the volume of a cuboid; calculate volumes and surface areas of cuboids.	2.2, 2.9-10
Construct, on paper and using ICT, a range of graphs and charts; identify which are most useful in the context of a problem.	1.23-25, 3.2, 3.7
Find and record all possible mutually exclusive outcomes for single events and two successive events in a systematic way.	1.5
Identify the necessary information to solve a problem; represent problems and interpret solutions in algebraic, geometric or graphical form.	various
Use logical argument to establish the truth of a statement.	various

Year 9

Add, subtract, multiply and divide fractions.	1.7-8
Use proportional reasoning to solve a problem, choosing the correct numbers to take as 100% or as a whole.	1.9-10
Make and justify estimates and approximations of calculations.	1.4, 2.15-16
Construct and solve linear equations with integer co-efficients, using an appropriate method.	1.18, 1.20
Generate terms of a sequence using term-to-term and position-to-term definitions of the sequence, on paper and using ICT; write an expression to describe the n^{th} term of an arithmetic sequence.	1.19, 3.7
Given values for m and c, find the gradient of lines given by equations of the form $y = mx + c$.	1.23
Construct functions arising from real-life problems and plot their corresponding graphs; interpret graphs arising from real situations.	1.24-25, 3.2
Solve geometrical problems using properties of angles, of parallel and intersecting lines, and of triangles and other polygons.	2.1, 2.4-5
Know that translations, rotations and reflections preserve length and angle and map objects onto congruent images.	2.12-13
Know and use the formulas for the circumference and area of a circle.	2.3
Design a survey or experiment to capture the necessary data from one or more sources; determine the sample size and degree of accuracy needed; design, trial and if necessary refine data collection sheets.	3.1
Communicate interpretations and results of a statistical enquiry using selected tables, graphs and diagrams in support.	3.2-3
Know that the sum of probabilities of all mutually exclusive outcomes is 1 and use this when solving problems.	3.5
Solve substantial problems by breaking them into simpler tasks, using a range of efficient techniques, methods and resources, including ICT; give solutions to an appropriate degree of accuracy.	1.4, 3.7, various
Present a concise, reasoned argument, using symbols, diagrams, graphs and related explanatory text.	various

Year 9 (extension)

Know and use the index laws for multiplication and division of positive integer powers.	1.14
Understand and use proportionality and calculate the result of any proportional change using multiplicative methods.	1.9-10
Square a linear expression and expand the product of two linear expressions of the form $x \pm n$; establish identities.	1.20-21
Solve a pair of simultaneous linear equations by eliminating one variable; link a graphical representation of an equation or a pair of equations to the algebraic solution.	1.22
Change the subject of a formula.	1.20
Know that if two 2-d shapes are similar, corresponding angles are equal and corresponding sides are in the same ratio.	2.12
Understand and apply Pythagoras' theorem.	2.7
Know from experience of constructing them that triangles given SSS, SAS, ASA or RHS are unique, but that triangles given SSA or AAA are not; apply these conditions to establish the congruence of triangles.	2.12
Use measures of speed and other compound measures to solve problems.	2.16
Identify possible sources of bias in a statistical enquiry and plan how to minimise it.	3.1
Examine critically the results of a statistical enquiry and justify choice of statistical representation in written presentations.	3.1-3
Generate fuller solutions to mathematical problems.	various
Recognise limitations on the accuracy of data and measurements.	1.4