Doing it with understanding

Colin Foster unpicks what we mean by teaching for understanding.

A mathematics teacher said to me this week that she did not just want her students to be able to "do it", but to "do it with understanding". She was talking about a particular mathematical procedure, but I think she would say the same thing about all topics and sub-topics in mathematics and for all of her classes. I have been thinking about what this might mean and reflecting on my own doing of mathematics. Am I a good model of "doing it with understanding"? Let's consider two examples.

1. What is 7 × 8?

When I hear this question, the words "fifty-six" come into my head instantly, and this feels more verbal than numerical. Of course, I know that "fifty-six" is a number, and I can say all sorts of things about that number, such as that it is between 50 and 60, but all of this seems to come afterwards. I have to think about that, whereas "fifty-six" is an immediate response, rather like responding to a phrase like "The proof is ..." with "... in the pudding", even if we perhaps do not know exactly why people say that, or what it is supposed to mean; it is familiar and sounds right. Only afterwards might we reflect on what it means. Does this mean that I am not answering 7 × 8 "with understanding"? Is it just a residue of "rote learning" from my childhood?

I can, of course, relate 7 × 8 = 56 to many other number facts and relationships if I want to. If you asked me how I knew it was true, I could come up with all kinds of justifications:

$$7 \times 8 = (7 \times 10) - (7 \times 2)$$
$$7 \times 8 = 7 \times 2 \times 2 \times 2$$
$$7 \times 8 = 8^2 - 8$$
$$7 \times 8 = 7^2 + 7$$
$$7 \times 8 = (5 \times 8) + (2 \times 8)$$

and so on. And I could represent these visually, using arrays or number lines or manipulatives. So does that mean that in some sense I do have good relational understanding (Skemp, 1976) of this fact? Maybe, and yet, to be honest, I am quite sure that none of this is going on in my head when I first say "fifty-six". It is all somehow available to me, but it is not present in my conscious mind. Is that bad?

Actually, I think it is good that my mind is not flooded with all of those connections and relationships every time I use this fact. The 7 × 8 = 56 is most likely part of a broader problem that I am working on and I want to save my precious, limited, working memory for thinking about wider aspects of what I am doing. I want the 7 × 8 = 56 to be automated, so as to free up my thinking for the more strategic, creative aspects of what I am doing. I do not want to get bogged down in a hundred ways of seeing why 7 × 8 = 56, unless that is somehow helpful to what I am doing. But then, does that mean that I am not "doing it with understanding"?

In some recent interviews with low-attaining 13-14-year-old students, I have been asking them to tell me how they know some of the number facts they know. How would they explain to someone else why they are correct? Often, they respond by saying that they would "do it in columns", and then write something like:

$$\begin{array}{r} 7 \\ \underline{\times\ 8} \\ \underline{56} \end{array}$$

pointing at the 7 and then the 8 and saying "seven times eight is … fifty … six" as they write it down. They claim that this is showing that it is true by "using the column method". To me, this may be an alternative way of writing the statement, but it does not provide me with any reason for believing that it is true. Surely,

$$\begin{array}{r} 7 \\ \underline{\times\ 8} \\ \underline{54} \end{array}$$

would look just as good, unless you already happen to know that that is wrong?

So, this sort of response does not convince me of relational understanding. However, I have found that, often, if I suggest models such as number lines or arrays, the students are able to construct perfectly good explanations to verify that they know a particular fact. Does this mean they do "really understand it", even though it was I who suggested the model? It is quite tricky to know how to interpret what is going on.

2. Rearranging equations

Rearrange this equation to make q the subject:

$$\frac{a}{d} = \frac{p}{q}$$

I can do this easily without writing anything down, as I am sure you can. I stare at the q and I just read off: "dp over a". How do I do it? I see the q move to where the a is and the other letters shuffle around. The d slides up next to the p and the a pops down to where the q was. It all happens almost instantly and the solution is staring me in the face. I suppose you could call this "cross-multiplying", but that is not a respectable term among teaching-for-understanding-oriented teachers, and it is not a rule that I am particularly keen on. I worry about students misapplying rules like this to situations such as:

$$\frac{a}{d} = \frac{p}{q} + 1$$

giving

$$aq = dp + 1$$

or:

$$\frac{a}{d} + \frac{p}{q} = 1$$

giving

$$aq + dp = 1,$$

which are both incorrect.

I also worry about students saying "this letter goes here" and "that one shoves over to there". I would pick up on this in class and ask them instead what *operations* they are doing to both sides of the equations, encouraging them to say that they are multiplying both sides of the equation by dq, to clear the fractions, and then dividing both sides by a. To me, that better supports understanding. And if you asked me to explain what I am doing when I give the answer "dp over a", that is what I would say. I would not talk about letters moving around. Does this mean I am inconsistent, secretly doing something myself that I would be concerned to see students doing?

It feels a bit wrong to be insisting on how others should be thinking about things if those ways of thinking are longer and harder than how I typically think about them. It is as though I have these handy shortcuts, which I find helpful and are perhaps the secret of my success, but I am unwilling to share them with my students. Perhaps students sometimes also have both a private way of doing things and a public, more respectable, version that they know they should offer when challenged? I often suspect this in classrooms when the teacher says, "How did you work it out?" and the student pauses and then replies, "Well, you could first do …", giving a long explanation with multiple steps. It seems highly unlikely to me that they could have done all that in the time that they took to give their answer. There is what you really do, and then there is what you say you did when asked.

I could go on with many more examples of quick or instant ways of working that I do. Should I be ashamed of these procedural shortcuts? Perhaps I should not be admitting in *MT* that this is what I do! But, I suspect that without these shortcuts, such as "just knowing" 7 × 8 and "just seeing" the letters move to where they end up when rearranging equations, I would be much slower and less accurate, and that would limit me when using these things in other parts of mathematics. I regard these as efficiencies that are underpinned by understanding, but I am not thinking about the reasoning behind them while I am doing the manipulations. The understanding is available if needed, but lies beneath the surface, not because I do not want to think, but because I *do* want to think, but not about the details of these small things; rather, I want to think at a broader, problem-solving level. I can think about these details when I want to, but I am not constrained to do so every time I use them.

Understanding what we are doing is of course critical, but I think we cannot forefront the understanding all the time, otherwise we would never get anything done. Think about how much it slows you down if you try to perform any everyday household task with a 3-year-old by your legs, constantly questioning why you are doing everything that you are doing! That kind of constant metacognitive questioning when doing mathematics would be highly disabling, and if we imply to our students that that is how they should be doing mathematics then I think we are making things much too difficult for them. If I operated like that, I would fail.

If we want our students to move from novices to experts, for want of better terms, I think we need to help them to automate common processes and not chastise them for "doing things without thinking about them". As Whitehead (1911) said, thinking about what you are doing is overrated:

> Civilization advances by extending the number of important operations which we can perform *without* thinking about them. (p. 61, emphasis added)

Otherwise, we clog up our students' brains thinking about how and why every detail works, and then they have no space to step back and keep track of the bigger picture of the mathematical problem that they are trying to solve.

If you insist on thinking about every detail of what you are doing, you can only ever do quite basic things. That is as true for you and me as it is for the most struggling mathematics student; the limits of working memory are a great equaliser across student and teacher. For example, if every time you replace 3 – (–1) with 3 + 1 you have to mentally go through some fiddly application of arrows on number lines, or some other model, it is going to pull you away from whatever you are doing that necessitates this; for example, solving the simultaneous equations:

$$x + 3y = 22$$
$$x - y = 6$$

Thinking in detail here about why $3y - (-y) = 4y$ is going to be a distraction from the main event and is going to work against you succeeding with the broader problem of solving these simultaneous equations.

I worry that in a quest to banish “rote learning” and “meaningless rules”, we sometimes end up effectively asking our students to prove everything every time they use it. For example, many people would add two fractions in this way:

$$\frac{2}{5} + \frac{3}{4} = \frac{(2 \times 4) + (3 \times 5)}{5 \times 4} = \frac{23}{20}$$

But I used to worry that this layout was too 'procedural'; too close to the “cross and smile” algorithm, and that students would always multiply the denominators, even when there was a lower common multiple. Instead I used to encourage students to write:

$$\frac{2}{5} + \frac{3}{4}$$
$$= \frac{8}{20} + \frac{15}{20}$$
$$= \frac{23}{20}$$

so that it was clearer, to me, at least, if not to them, that we were simply making two fractions equivalent to the original two fractions, but with the same denominators. I felt that this vertical layout, and writing the common denominator twice, was a more “teaching for understanding” way to do it. But does this just load students down with effectively having to explain to themselves what they are doing at the same time as doing it, and is this perhaps counterproductive?

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