Mathematical white lies

Colin Foster and Mike Ollerton discuss the complexities of making `always-true´ mathematical statements in the classroom.

Should we tell lies to children? Of course not. Should we tell them Santa Claus exists and then later, when they are a bit older, explain it was "just a white lie"? What about white lies in mathematics? Should we cross our fingers behind our backs and say, "You can't take away 4 from 3", or should we try to always tell the truth, the whole truth and nothing but the truth? Or is this unrealistic and too much to ask?

We do not think the question "Can you take away 4 from 3?" has one right answer. The answer could be "No" or "Yes", depending on the context. In lots of contexts with which children are familiar, involving physical objects, we think it is quite reasonable to say "No". We do not think this is 'a misconception' or necessarily betrays lack of awareness of negative numbers. It might just mean that negative numbers are not seen to be relevant here. We do not think you can take away 4 apples from 3 apples, even though we know about negative numbers, so we would prefer to say that "You can't take away 4 from 3" in this context, rather than claim that "Mathematically, it means you actually owe one apple", which seems an odd idea; whoever owes apples?! Furthermore, the two questions: "What is the difference between 3 and 4?" and "What is the difference between 4 and 3?" both have the same answer.

For us, the issue is both about making choices and context. If we want to work on the natural numbers, then 3 – 4 has no answer. If we want to work on all the integers, then 3 – 4 = –1; it depends on the domain. We do not think 'domain' has to be an advanced concept that students meet only at GCSE/A-level. Young children are very used to things being true or not true, depending on context; this is not necessarily a hard idea for children to grapple with. If I am in a lift on the third floor, can I go down four floors or not? It depends on whether there is a basement. A similar complexity arises with regard to how many floors and how many stories a house might have. As such, in the abstract, lots of questions could be answered, "It depends".

So, we would not want to criticise a teacher who says, perhaps in the context of column subtraction, that 3 – 4 "can't be done". We do not think that this is necessarily false or 'backward-facing' (McCourt, 2019, p. 115), or is creating a 'misconception' about negative numbers that must be 'undone' later. The subtraction cannot be done on the natural numbers, which might be the assumption behind the way column subtraction is being performed.

However, it *is* possible to use negative integers when doing column subtraction; for example:

Tens Ones

$$\begin{array}{rcc} & 5 & 3 \\ - & 2 & 4 \\ \hline & 3 & {}^{-}1 \end{array} \quad \text{and } 30 + {}^{-}1 = \underline{29}$$

In some ways, maybe this is preferable to the more usual 'borrowing' of a 10:

$$\begin{array}{rcc} & {}^{4}\not{5} & {}^{1}3 \\ - & 2 & 4 \\ \hline & 2 & 9 \end{array}$$

With 'borrowing', for the '13', we effectively write *two* digits in a single column, when normally only one digit is allowed. So, in these kinds of subtractions, the choice is either to break the 'natural numbers only' rule or to break the 'one digit per column' rule. There are pros and cons. However, if we are going to use the latter method, might it not be perfectly correct to say, in this context, that 3 – 4 "can't be done"?

Another example might be: "Can you square root a negative number?" If children are just learning about square roots for the first time, and they are exploring, perhaps with calculators, then many issues might arise: the square roots of square numbers are positive integers; the square roots of non-square integers are irrational, and produce non-repeating, non-terminating decimals. But the square roots of *some* decimal numbers are *rational,* e.g. $\sqrt{2.25}$. Square-rooting a negative number is going to give an error on the calculator, meaning that, as far as the calculator is concerned, "You can't square root a negative number". So, does this mean it is OK for the teacher to say that? If 'square roots' and 'irrational numbers' and 'non-repeating, non-terminating decimals' are all new ideas today, do we also want 'imaginary numbers' to be yet another new concept? (Furthermore, might 'irrational' and 'imaginary', both beginning with 'i', then get muddled up?) The fact that two negative numbers multiply to make a positive number (just as two positive numbers do) could also be a shaky idea, so might raising the prospect of imaginary numbers be a step too far? Might it not be preferable to organise a discussion about the question: "Can you take the square root of a negative number?", and let students think about that. One possibility could be to say something like, "The square root of a negative number isn't a real number", but if the students do not know that 'a real number' is a technical term, then they may just hear this as "The square root of a negative number isn't really a number" and interpret this as "The square root of a negative number doesn't exist". So, we are not sure that this actually gets us off the hook. It is hard to say what 'a real number' means without some sense of the possibility of numbers that are *not* real.

We think the answers to many mathematical questions depend on what our domain is. Many statements, therefore, are neither true nor false in isolation; it depends on the context:

- To multiply by 10, you just place a zero on the right-hand side: (This might appear to be what happens for integers, though is not so good for decimals or fractions).
- Multiplying makes things bigger: (This works in some situations students meet, though not for a calculation such as $\frac{1}{2}$ x $\frac{1}{2}$, or, indeed, for any number multiplied by any other number less than 1).

(See Dougherty, Bush and Karp [2017] for more examples of 'rules that expire'.)

If we want every statement that we make in the classroom to be absolutely and completely true, from all perspectives, for all situations a student is ever going to encounter in the future, then we might be making life much too difficult for both ourselves and our students. We would either be too scared to say anything or would have to introduce numerous caveats, which would be meaningless to anyone without more advanced knowledge than the concepts students are currently learning. Much elementary mathematics depends on assumptions or axioms that would be too complicated to set out fully. A university mathematics lecturer once said, "This statement is true for 'nice' functions, but defining what I mean by 'nice' would be a whole course in itself"! We cannot perfectly futureproof all of our teaching.

Perhaps not much is *always* true regardless of *any* assumptions or context. Perhaps, instead of criticising things for being partially true, we might choose to accept that partially-true statements may often be the best we can do. We can subsequently focus students' attention on exploring under what conditions certain statements are always, sometimes or never true, remembering that a statement being 'never' or 'always' true will depend on what kinds of numbers or other mathematical objects students are currently aware of, or have an 'at-homeness' with. (See Cockcroft, 1982, para 39: "We would wish the word 'numerate' to imply the possession of two attributes. The first of these is an 'at-homeness' with numbers and an ability to make use of mathematical skills which enables an individual to cope with the practical mathematical demands of his (*sic*) everyday life".)

References

Dougherty, B. J., Bush, S. B., and Karp, K. S. (2017). Circumventing High School rules that expire. *Mathematics Teacher*, *111*(2), 134-139.

Cockcroft Report (1982). *Mathematics Counts: Report of the Committee of Inquiry into the Teaching of Mathematics in Schools under the Chairmanship of Dr WH Cockcroft.* London: Her Majesty's Stationery Office.

McCourt, M. (2019). *Teaching for Mastery.* Woodbridge, UK: John Catt Educational Ltd.

Colin Foster is a Reader in Mathematics Education in the Mathematics Education Centre at Loughborough University. He can be contacted at c@foster77.co.uk www.foster77.co.uk @colinfoster77

Mike Ollerton is an almost-retired freelance consultant still passionate about mathematics education. mikeollerton@btinternet.com www.mikeollerton.com @MichaelOllerton