Empty Number Lines for Solving Linear Equations

By **Chris Shore, Colin Foster and Tom Francome**

In this article, we explore the possibilities associated with using empty number lines for teaching the solving of linear equations. In our recent work designing the *LUMEN Curriculum* (www.lboro.ac.uk/services/lumen/curriculum/, Foster et al., 2024), we have been seeking coherence across the different representations that we have used. We have adopted a 'less is more' approach to representations and models, choosing to prioritise the use of the number line over other representations wherever possible (Foster, 2022, 2023). This principle presents us with frequent design challenges, and here we explore the boundaries and limitations of the use of empty number lines within one important area of mathematics content – solving linear equations.

Balancing representations

Many different representations are commonly used with school students to visualise the process of solving a linear equation such as $3x + 7 = 19$. The rationale of 'Do the same operation to both sides' is often conveniently displayed using the language and imagery of 'balancing' (Andrews & Sayers, 2012; Marschall & Andrews, 2015; Otten et al., 2019). However, reservations have been expressed about modern children's lack of familiarity with two-pan balances (Fig. 1). Even playground seesaws nowadays tend to be constructed with springs (Fig. 2), so they no longer function in the straightforward up-and-down fashion of a balance.

Figure 1. A traditional two-pan balance – a common metaphor for equation solving

Figure 2. A modern seesaw in a playground

Additionally, where solving a linear equation involves steps such as expanding brackets and collecting like terms, the metaphor of balancing can become strained, because although these steps maintain equality, they may happen on only one side of the equation. This leads us to question how useful a traditional balance model may be as a metaphor for equation solving. In particular, given that there is a cost associated with introducing every additional representation (Foster, 2022), and our intention to adopt the number line as our default representation whenever possible, we wonder what possibilities an empty number line offers, as an alternative to balances, for representing the solving of linear equations.

Solving equations on the Empty Number Line

Consider the two layouts shown in Figures 3 and 4 that might be employed for solving the simple linear equation $3x + 7 = 19$. Figure 3 uses a traditional vertical layout and Figure 4 an unscaled, horizontal empty number line (Dickinson & Eade, 2004). Learners at this stage may bring familiarity with using empty number lines for numerical calculations; we would not envisage solving linear equations being the first time that learners met empty number lines.

$$\begin{aligned} 3x + 7 &= 19 \\ \downarrow -7 \quad & \quad \downarrow -7 \\ 3x &= 12 \\ \downarrow \div 3 \quad & \quad \downarrow \div 3 \\ x &= 4 \end{aligned}$$

Figure 3. Solving $3x + 7 = 19$ using a traditional vertical layout

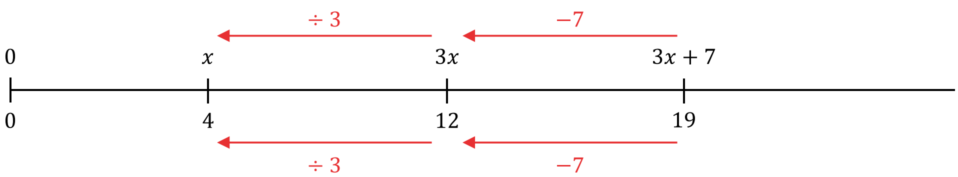

Figure 4. Solving $3x + 7 = 19$ using a horizontal empty number line.

The similarities between the layouts in Figures 3 and 4 are notable, with pairs of identical operations being applied successively to 'both sides' of the equation to reach the solution, $x = 4$. In both layouts, we have used arrows to indicate operations, and it would be possible to differentiate addition/subtraction and multiplication/division operations by using different colours or types of arrows. Since arrows are ubiquitous throughout school mathematics, such a distinction might help to avoid confusion between correct and incorrect simplifications, such as those shown in Figure 5.

Figure 5. Correct / incorrect ratio simplifications

It might be thought that the two representations in Figures 3 and 4 are essentially identical, but for a 90° rotation. A vertical number line provides more breathing space for writing longer expressions horizontally and for completing more steps vertically 'down the page' (Fig. 6). It also supports the language of 'left-hand side' and 'right-hand side'. We think that vertical number lines have considerable merit (Moeller et al., 2025), but for convenience in this article we will use more familiar horizontal number lines.

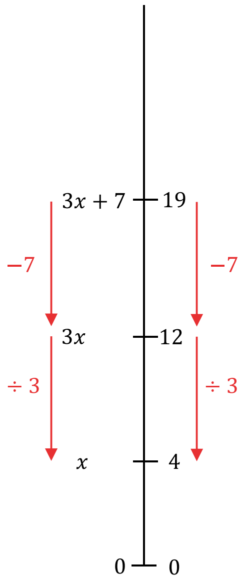

Figure 6. Solving $3x + 7 = 19$ using a vertical empty number line provides more breathing space for writing longer expressions

As we have contrasted Figures 3 and 4, we have come to think that the differences between these two approaches are highly significant, with considerable implications for the learning of algebra, as we will explain here.

The traditional vertical layout shown in Figure 3, if interpreted as a solution to the equation at the top, has to be read or drawn from top to bottom, and Figure 4, employing the empty number line, has to be read from right to left. The vertical progression down the page in Figure 3 has no more significance than the progression of ordinary text down the page; however, the horizontal positions of the six expressions in Figure 4, although not to scale, are *very* significant. In this example, all is well: not only are the pairwise coincidences ($3x + 7$ with 19, $3x$ with 12, and x with 4) essential for expressing the three equalities, but the numbers underneath the line $(0, 4, 12, 19)$ are correctly *ordered*, expressing the relative magnitudes of these quantities; i.e.,

$$0 < 4 < 12 < 19 \text{ and } x < 3x < 3x + 7.$$

Mis-ordering on the Empty Number Line

It follows from what we have seen that where the necessary operations to solve the equation *increase* the value of both sides, movement to the *right* is needed (e.g. Figs. 7 and 8). In cases such as Figure 7, where movement is non-unidirectional, ordering problems can arise. For example, if, instead of solving $3x - 7 = 8$, as in Figure 7, we were to tackle the similar-looking equation $3x - 7 = 2$, the same layout results in breaking the order $0 < 2 < 3 < 9$ (Fig. 9). (The same problem would arise in Figure 7 if we had chosen to make the $\div 3$ arrow shorter than the $+7$ arrow.) Given that empty number lines are typically not drawn to scale, losing order as well as magnitude means that the line has completely ceased to represent any feature of the numerical values. While an expert might be happy to overlook this, or even be oblivious to it, we feel that it is the kind of inconsistency that a thoughtful learner would puzzle over.

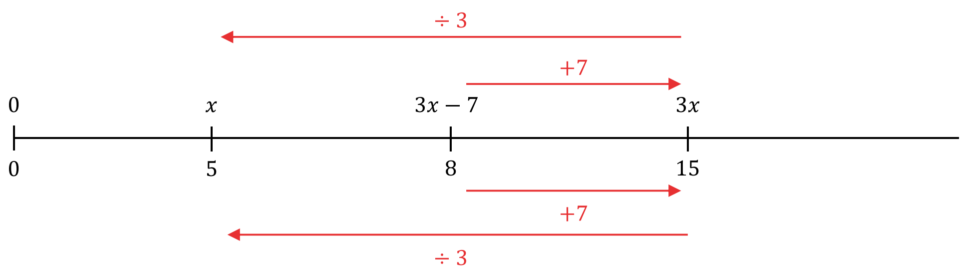

Figure 7. Movement to the right in the solution of $3x - 7 = 8$

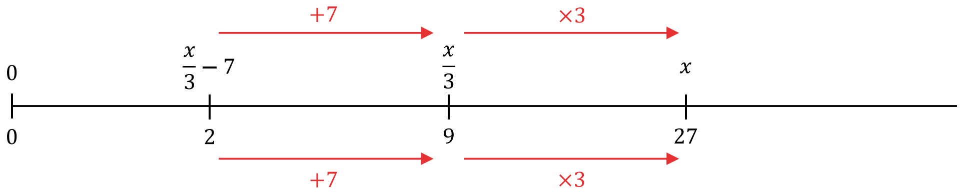

Figure 8. Movement to the right first in the solution of $\frac{x}{3}-7=2$

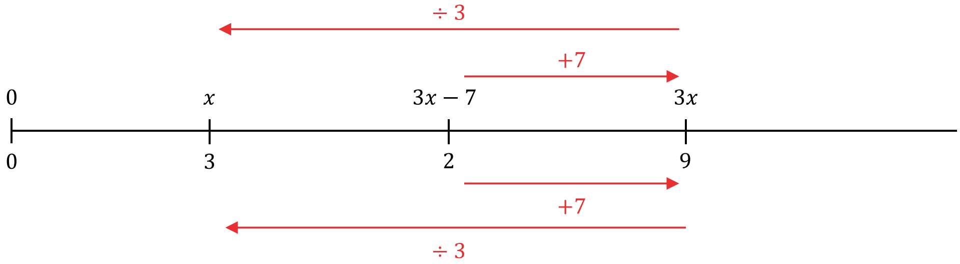

Figure 9. Non-unidirectional movement breaking the order $0<2<3<9$ when solving $3x-7=2$

Adopting the empty number line approach and preserving ordering on the number line forces the learner to consider with each operation whether the value of both sides increases or decreases, and by how much, relative to the other quantities. In simple cases, such as $+\ 5$ or $\times\ 3$ applied to positive values, this may present little challenge, and the effort may be compensated for by the benefit of visualising the 'changing but remaining equal' nature of the values at each stage. However, in other cases, this could be an unwelcome distraction, unhelpfully increasing learners' cognitive load and making calculational demands that would otherwise be unnecessary. In some cases it could even require knowing in advance the unknown value, and whether it is positive or negative. For example, when solving $3x-7=2$, to preserve the order of the empty number line, the learner needs to *anticipate* the solution $x=3$ before completing the final step, and then squeeze this value in between the 2 and the 9 (Fig. 10).

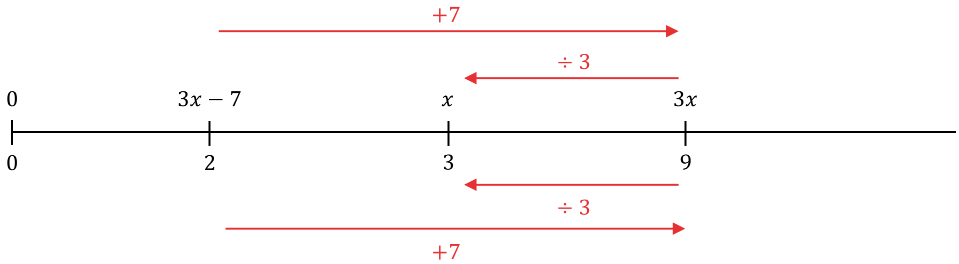

Figure 10. Preserving ordering can be challenging when movement is non-unidirectional

We think that there could be other benefits to discussing the ordering of values along with movements on an empty number line; for example, when addressing misconceptions such as 'multiplication makes things bigger'. Representing $\times\ 0.1$, for example, as a movement to the left can help learners make sense of the operation beyond merely seeing movement of the digits one column to the right. We think that much of this work can be undertaken within 'number' before encountering algebraic manipulation.

In more subtle cases, the mis-ordering may not be apparent unless the learner goes back after the solution is obtained and checks for it. For example, in solving $4x-7=x+8$, the representation in Figure 11 may at first glance look innocent. However, having discovered at the end that $x=5$, the order of the expressions turns out to be 0, 5, 15, 13, 20, with $x+8$ out of order. This occurs because, when subtracting the x, we drew this as though $x>7$, but, as we may or may not realise afterwards, this was false. This is a frequent problem when adding or subtracting multiples of the unknown, and a further problem relates to the necessity of making an assumption about the *sign* of x, in order to decide in which *direction* a $+x$ or $-x$ arrow should go.

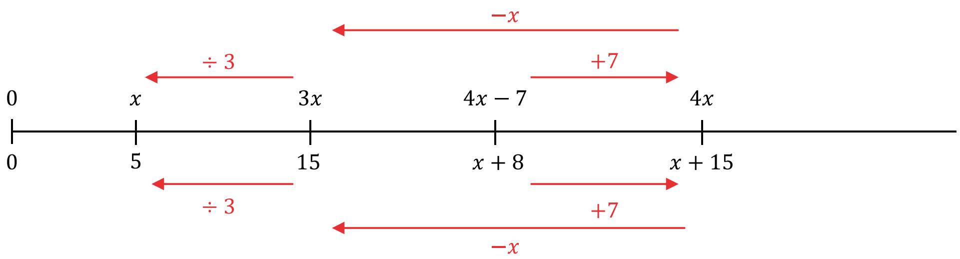

Figure 11. Solving $4x-7=x+8$ using the empty number line

One way to avoid this issue is always to take the equations back to zero. In this case, we would subtract 8 and then subtract x, to leave $3x - 15 = 0$. Adding 15 and dividing by 3 is now unproblematic. We can always know where *zero* is, even on an empty number line. This approach can forshadow work with higher-order polynomials, such as quadratics, where obtaining an expression equal to zero is the prelude to factorising and using the zero-product property.

Weighing advantages and disadvantages

In our design discussions, we have debated how important or not the mis-ordering might be in cases such as the one shown in Figure 11. All models have their limitations and break down eventually, and then need adapting, extending or abandoning. Have we just pushed this model too far? Our reservations over the traditional layout of Figure 3 are that it is completely symbolic and abstract, whereas the almost identical layout in Figure 4 or Figure 6 seems to us to root each stage at a particular location on the empty number line. To what extent is this helpful or unhelpful for the learner? It may help the learner to be reminded that, at every point, each side of the equation is an expression that has a particular value – and, at each point, the values of each expression are the same on each side. Before *solving* equations, we would envisage learners having had many prior experiences starting with, for example, $x = 3$, and devising other equalities that are consistent with this one, building up more and more complicated-looking equations.

But should a learner be concerned at each step of solving an equation about the value that both sides of the equation have? Does this add unnecessary cognitive load, or does it support sensemaking of what they are doing? We aspire to an eventual expert performance in which the learner may not think or care about whether the value is increasing or decreasing in each step of their solution, confident only in the fact that both sides remain equal to each other, even if they are unaware of what particular value they are equal to. Is this a more advanced perspective or a less conceptual one? Perhaps the empty number line might be deployed initially, in carefully-chosen cases in which the ordering problems do not arise, and then, subsequently, this scaffolding can be removed, and learners proceed purely abstractly, using a traditional layout as in Figure 3. If so, would the teacher ever have good reason to draw attention to the ordering problem or not?

The empty number line approach emphasises not only that the same operations are performed on both sides of the equation but that the 'value of the equation' *changes* throughout the process, while *preserving* the solution set of the unknown. The idea that the values of both sides of the equation are changing with each step, but in such a way that they always remain equal to each other, may not always be appreciated clearly by learners. This is perhaps suggested when learners include additional (incorrect) equals signs down the left-hand side of their solution (Fig. 12), appearing not to distinguish a situation such as Fig. 12a (incorrect) from Figure 12b (correct).

(a)	(b)
$3x + 7 = 19$	$3x + 7 - 19$
$= 3x \quad = 12$	$= 3x - 12$
$= x \quad = 4$	$= 3(x - 4)$

Figure 12. Equals signs on the left-hand side used (a) incorrectly and (b) correctly

The empty number line approach perhaps makes it easier for learners to talk about an equation such as $2x = 6$ being, in some sense, 'twice' an equation such as $x = 3$, despite the value of x being the same in both. We wonder if this is useful or misleading language? We think that this way of talking can be convenient when learners progress to solving simultaneous equations and want to label an equation such as $x + y = 3$ as ① and a scaled-up version, such as $2x + 2y = 6$, as $2 \times$ ①. Perhaps it could be worth living with some of the problems outlined with the empty number line approach if the benefits when extending to simultaneous linear equations were considerable enough.

Simultaneous linear equations

Since representational coherence across different topics is central to our design principles, it is important for us to consider how effective the empty number line representation might be in related algebra content beyond solving simple linear equations, such as for solving simultaneous linear equations. We want to avoid introducing multiple, bespoke representations that have limited domains of relevance, as we see this as contrary to the coherence that we wish to build into our resources.

The empty number line can be used to solve a pair of simultaneous linear equations in two unknowns by repeatedly finding operations that translate both equations (note, *equations* now, rather than expressions) to the same position. Fig. 13 shows how $x + 2y = 8$ and $2x + 3y = 13$ may be solved by transforming *both* equations into $2x + 4y = 16$. We multiply $x + 2y = 8$ so that the coefficient of x matches the corresponding coefficient in $2x + 3y = 13$. Then, $2x + 3y = 13$ needs 3 adding to the 13 and y adding to the $2x + 3y$ to reach the same position on the number line, meaning that $y = 3$.

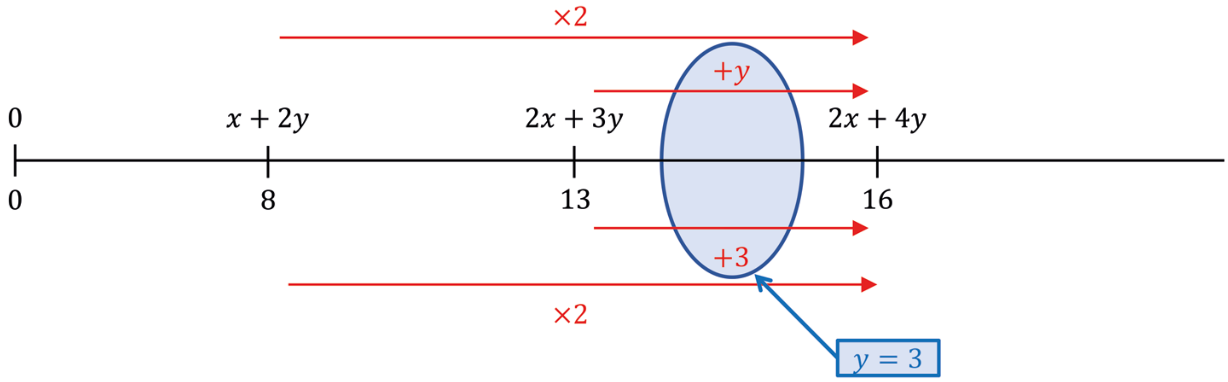

Figure 13. Solving the simultaneous equations $x + 2y = 8$ and $2x + 3y = 13$ on the empty number line

It is worth noting that now the arrows above and below the number line show *different* symbols, yet their value is the same. We might regard this as the same representation (empty number line) but a different model. This way of working with simultaneous linear equations might perhaps be less problematic for learners if they have previously employed the same kind of empty-number-line approach with simple linear equations. Figure 14 shows the solution of the same equation as in Figure 11, but using the empty number line differently. Here, we make both the left-hand side and the right-hand side of the equation $4x - 7 = x + 8$ into the same expression. This requires adding 15 to the left-hand side and $3x$ to the right-hand side, and so it follows that these two quantities must be equal. The fact that we are still 'doing the same thing to both sides' is less transparent here, but we use this fact to equate 15 and $3x$. (To obtain $x = 5$ from $3x = 15$, we could draw a new empty number line and solve by dividing by 3, although we envisage learners soon not needing this additional number line for single-step equations.)

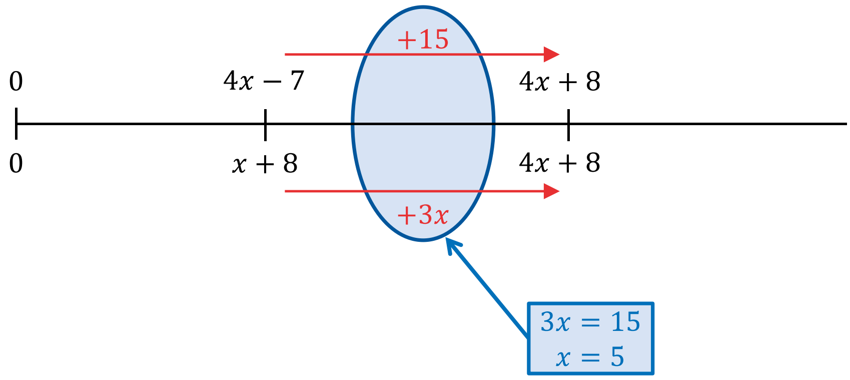

Figure 14. Solving the simple equation $4x - 7 = x + 8$ in a different way on the empty number line

Quadratic equations

If we now turn to the solution of quadratic equations, we note that solution by completing the square can be accomplished step by step on the empty number line. Figure 15 shows the solution of the equation $x^2 + 4x - 5 = 0$ from the point at which it is re-written in the form $(x + 2)^2 - 9 = 0$. We think that solution by factorisation cannot reasonably be represented on the empty number line, unless you branch off to two different number lines. Cartesian axes (i.e. two orthogonal number lines) seem necessary here. However, using an empty number line to help learners generalise that for $ax + b = 0$ we have $x = -\frac{b}{a}$ can help when finding solutions for quadratics in factorised form.

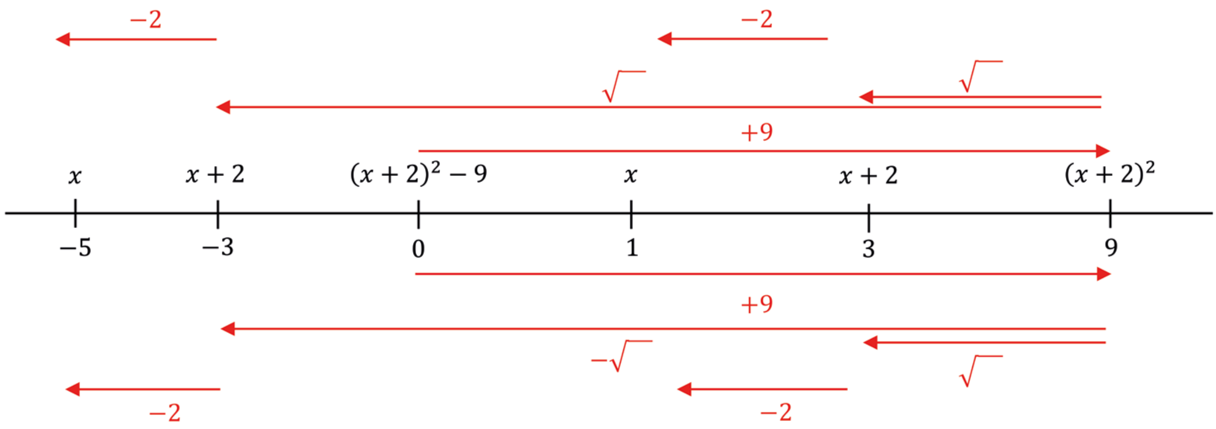

Figure 15. Solving $(x + 2)^2 - 9 = 0$ by completing the square

Multiplication by –1

For equations in which the coefficient of the unknown is negative, multiplication or division by –1 may be useful, and is represented by rotation by a half turn about zero. In 1 dimension, this rotation is equivalent to a reflection in zero. However, we prefer to frame this as rotation, because we see this as more forward looking. Eventually, when learners meet complex numbers, they will encounter i as a rotation of a *quarter* turn anticlockwise about zero. Two of these quarter rotations, successively, are equivalent to this half-turn rotation about zero, and this corresponds to $i^2 = -1$. Although complex numbers may be years away in terms of learners' journeys, we see no reason not to prepare the ground in this way.

It is always possible to avoid multiplication by a negative number. For example, Figure 16 shows solving $7 - 3x = 4$, in which the first step adds $3x$, rendering a positive coefficient of x.

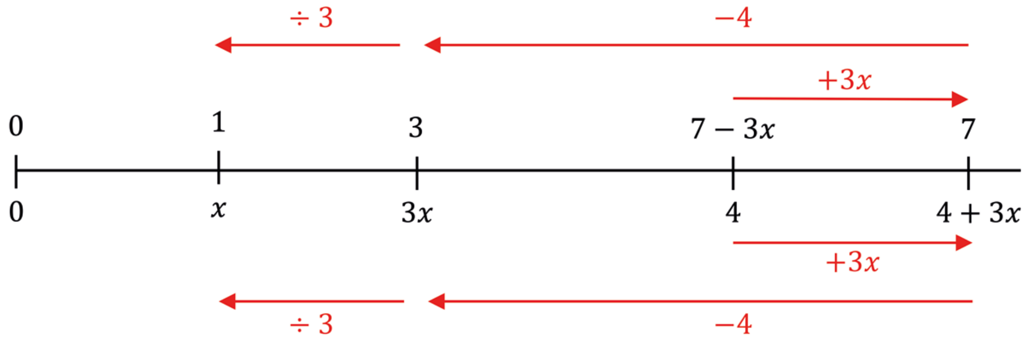

Figure 16. Solving $7 - 3x = 4$ by adding $3x$ first

Alternatively, Figures 17 and 18 show two ways in which division or multiplication by a negative number can be visualised on the empty number line. In Figure 17, we use multiplication by -1 to rotate $-3x$ into $+3x$. Whereas the approach in Figure 16 switched the term in the unknown from above to below the number line, multiplication by a negative number keeps the term in the unknown above the number line. In Figure 18, we save a step by dividing by -3 instead.

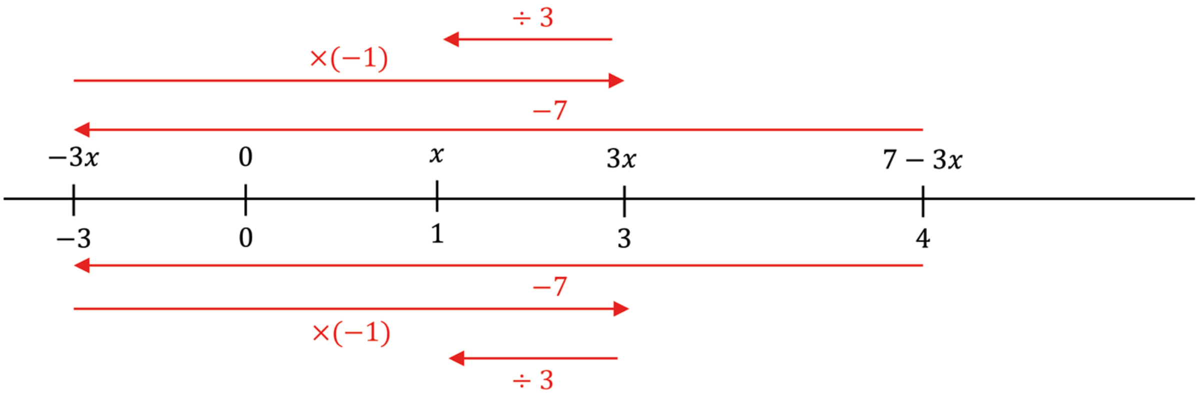

Figure 17. Solving $7 - 3x = 4$ by multiplying by -1

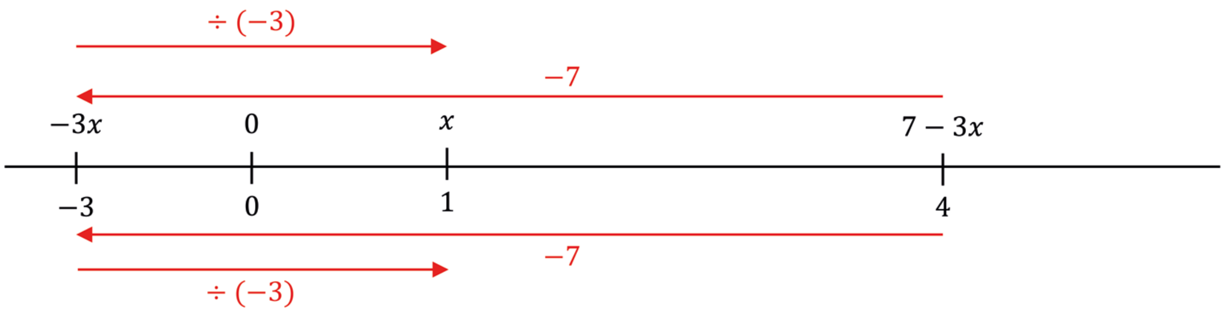

Figure 18. Solving $7 - 3x = 4$ in two steps by dividing by – 3

Simple linear inequalities

Our biggest disappointment in our adventures with the empty number line has been our failure to deploy them for solving simple linear inequalities. Learners are often asked to represent the *solutions* to inequalities on a number line, such as the solution set $x \leq 4$ to the inequality $3x + 7 \leq 19$ in the form shown in Figure 19. This raised our hopes that the empty number line could be ideal for representing the entire process of solving inequalities.

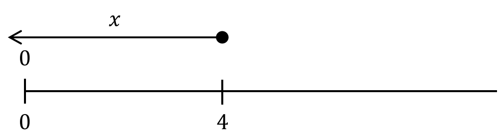

Figure 19. Representing the solution $x \leq 4$ to the inequality $3x + 7 \leq 19$ on the empty number line

However, this seems possible only under highly restricted circumstances, such as the example in Figure 20 for the inequality $3x + 7 \leq 19$. In situations in which there is an unknown on both sides (e.g. $4x - 7 \leq x + 8$), the symmetry between the two instances of the unknown must be broken, as shown in Figure 21. Here, the $x + 8$ is treated as a 'fixed' value, marked at a single position on the number line, while the $4x - 7$ is represented by a continuous arrow. The roles of the two sides could of course be reversed, but we cannot see how to represent *both* sides simultaneously as arrows.

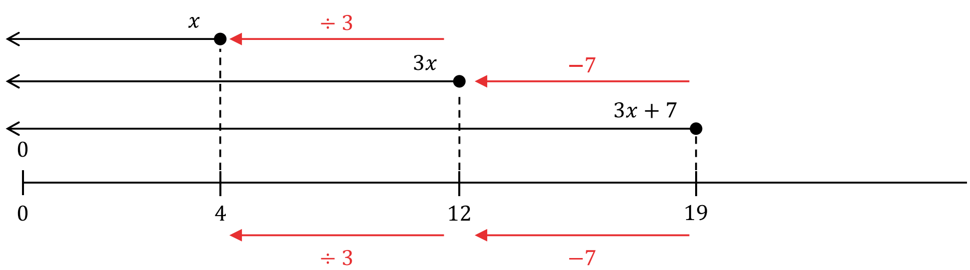

Figure 20. Solving the inequality $3x + 7 \leq 19$ on the empty number line

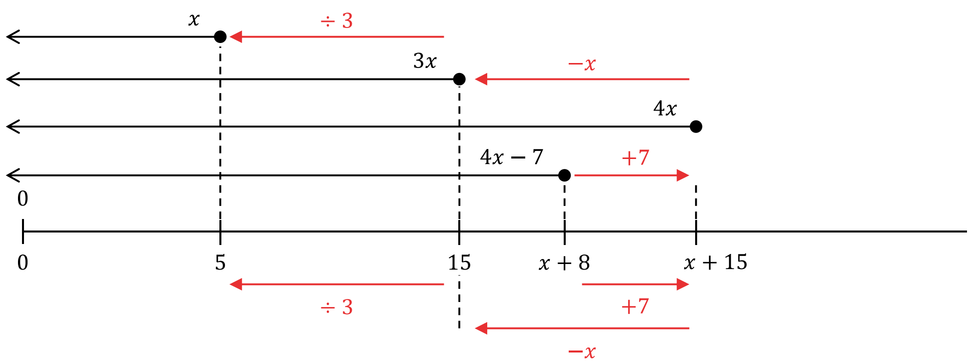

Figure 21. Solving the inequality $4x - 7 \leq x + 8$ on the empty number line

We also regret the possible confusion of arrows representing ranges of values and arrows representing operations being performed on both sides. This might be mitigated by using different arrow styles or colours. As we suggested earlier, it may also be easier here to reduce the inequality to one that has a zero on one side.

Double inequalities can be straightforward if they are of the form $a \leq x \pm b \leq c$, where a, b and c are constants (Fig. 22), but in other cases they can also be complicated to represent.

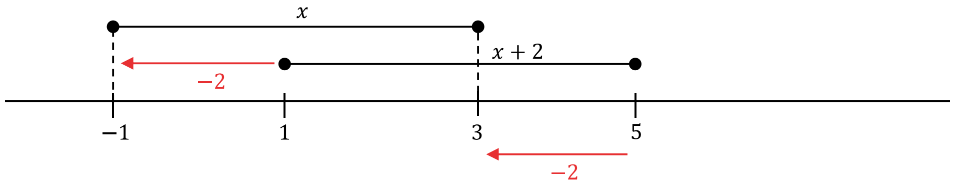

Figure 22. Solving the double inequality $1 \leq x + 2 \leq 5$ on the empty number line

Conclusion

Given our ambition of trying to limit the proliferation of multiple contrasting representations throughout the mathematics curriculum (Foster, 2022), it seems to us productive, before selecting a representation for use in some content area, to explore how well or badly it might extend to other related areas. We have sought to explore that in this article for the case of the empty number line to see what opportunities and challenges there might be in applying this as the principal representation for solving equations (and inequalities). We have examined how the empty number line might be implemented for solving simple linear equations with the unknown on both sides, simultaneous linear equations, quadratic equations, simple linear inequalities and double inequalities.

Our tentative conclusions are that we do not think that the empty number line can be a desirable end-goal for learners' layout of their solutions in these topics. To do this accurately and consistently is cumbersome and sometimes requires knowing the solution in advance. But we think that the empty number line may offer a helpful scaffold in the early stages of learning to solve simple equations, providing possibly clearer links to prior number concepts than the balance model seems likely to do. But we would envisage this scaffolding being faded away before learners encounter examples that possess many of the kinds of complexities discussed in this article.

It may also be that algebra is an important enough topic to justify introducing an alternative bespoke representation, such as *Grid Algebra* (Hewitt, 2016). *Grid Algebra* is based on the multiplication table grid, and can be seen as multiple, 'stacked' number lines. All representations and models have costs as well as benefits, and careful thought needs to be given to whether the benefits of introducing a particular model outweigh the costs and, if so, for how long this remains the case.

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