The Sheer Delight of Shearing

By **Chris Shore, Tom Francome and Colin Foster**

When students think about area, such as the area of a triangle, they often go straight to a formula, such as Area $= \frac{1}{2} \times \text{base} \times \text{height}$. Giving students questions in which this formula cannot be immediately applied, because the lengths of the base and the height are not explicitly provided, may encourage them to think more deeply about area.

The triangle shown in Figure 1 has all of its vertices on lattice points. How would you work out its area?

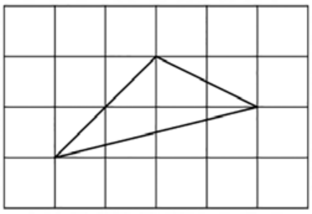

Figure 1. A triangle with all of its vertices on lattice points.

Maybe you would use Pick's Theorem? This tells us that the area of any simple polygon with all of its vertices on lattice points is equal to $\frac{b}{2} + i - 1$, where b is the number of lattice points on the boundary and i is the number of interior lattice points (i.e., *not* on a boundary) (see [Scott, 2006] for a proof). Here, $b = 4$ and $i = 2$, so the area must be 3.

Or maybe you would use Pythagoras' Theorem to find the lengths of the sides and then use Heron's formula to find the area from the three side lengths. From Pythagoras' Theorem, the three sides are $2\sqrt{2}$, $\sqrt{5}$ and $\sqrt{17}$. Substituting these values into Heron's formula,

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)},$$

where a, b and c are the lengths of the three sides, and s is the semi-perimeter gives a rather formidable expression. Somewhat miraculously, it does indeed simplify to 3, although it would be fair to say that this is not the easiest way to find the area!

It would even be possible to find the area by using the more heavy-duty machinery of trigonometry and formulae such as Area $= \frac{1}{2} ab \sin C$. Another option would be to calculate half of the magnitude of the cross product of two vectors representing two of the sides; e.g., $\frac{1}{2}\left|\begin{pmatrix}2\\2\end{pmatrix} \times \begin{pmatrix}2\\1\end{pmatrix}\right|$. Equivalently, we could assign coordinates to the three points – i.e., (x_1, y_1), (x_2, y_2) and (x_3, y_3) – and then use the formula:

$$\text{Area} = \left|\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|.$$

There are clearly numerous ways to approach this question.

Simple approaches

It is interesting to think about these different possibilities, but how might you expect a lower-secondary school student to go about finding the area of the triangle given in Figure 1 (Note 1)? The most common way that we have seen taught for this kind of question is to enclose the triangle in the smallest possible rectangle that has its sides parallel to the grid lines, and subtract the surrounding right-angled triangles (Figure 2):

$$4 \times 2 - \frac{1}{2}(1 \times 2) - \frac{1}{2}(1 \times 4) - \frac{1}{2}(2 \times 2) = 3$$

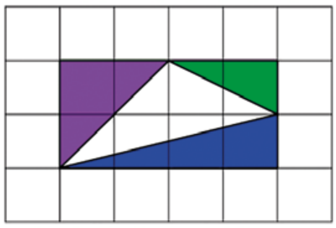

Figure 2. Finding the area of the central triangle by enclosing it in a rectangle and subtracting three (coloured) right-angled triangles.

This approach perhaps assumes that the easiest areas to calculate are rectangles and right-angled triangles, but maybe this depends on how the area of polygons is built up. It is usual to begin with a rectangle and think about area by counting up rows and columns of squares that fill it. This leads to finding this number by multiplying two perpendicular lengths. Then come right-angled triangles, as half of a rectangle. This might begin with *isosceles* right-angled triangles, where every part-square on the grid is exactly *half* of a square – and then progress to the harder case where every part-square has a partner that completes it into a square (shown by the colour in Figure 3). When both the base and the height of the right-angled triangle are odd, this doesn't quite work, because we end up with an odd number of half squares, leading to a leftover half square at the end (e.g., Figure 4).

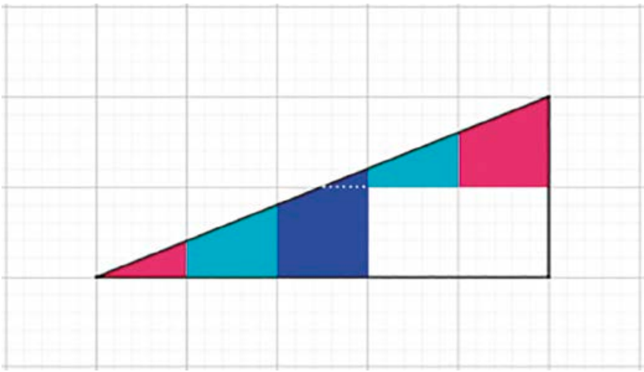

Figure 3. Every part-square pairs up with another part-square to make a whole.

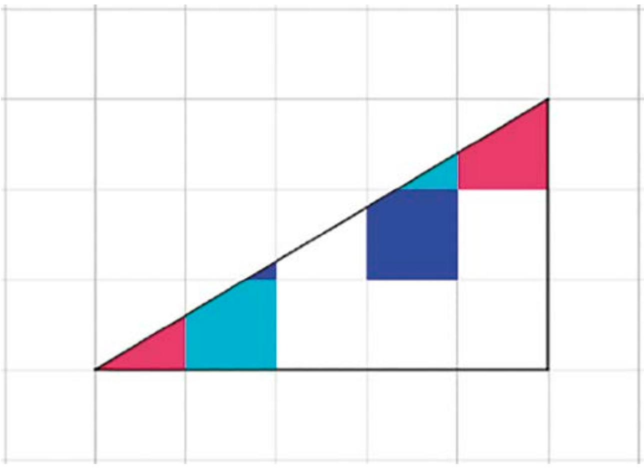

Figure 4. Here, part-squares pair up to make whole squares, but there is a half square left over at the end (shown in white).

To see *non*-right-angled triangles also as half of a rectangle usually entails dividing the triangle into two right-angled triangles, and seeing each of them as half of a different rectangle (e.g., the green and blue rectangles shown in Figure 5). This requires somehow seeing (or assuming) that the sum of the halves of two rectangles is equal to half of the sum of the rectangles.

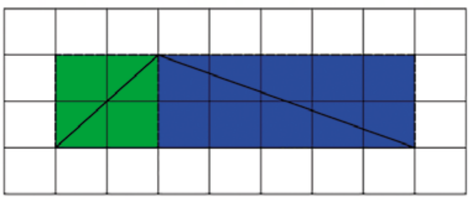

Figure 5. Seeing a non-right-angled triangle as half of the large (green + blue) rectangle.

Shearing

An alternative approach is to bring in the idea of *shearing* early on (Foster, 2019). Shearing is a very useful area-preserving transformation, and can be a helpful way to see why 'perpendicular height' is the relevant measure to consider when dealing with parallelograms and non-right-angled triangles. If learners can find the area of a rectangle, then they can find the area of *any parallelogram*, because if neither the base nor the height changes as we shear a parallelogram into another one, then the area must be constant. Pushing over a stack of paper or books is a classic way of seeing this (Figure 6), and this can be easier than the area dissection proofs that are more commonly used. This means that the 'base × height' formula for a rectangle is valid not just for rectangles but for *any* parallelogram (of which rectangles are merely a subset), so long as we interpret 'height' as 'perpendicular height'. This seems like an important insight, and we think that there may be a case for meeting this early, before worrying about triangles.

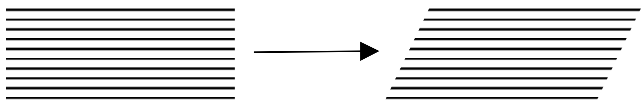

Figure 6. Pushing over a stack of books indicates that shearing preserves area.

Once students are convinced about shearing, then the nice thing is that *every* triangle is half of a parallelogram. Right-angled triangles are no longer a particularly special case – the parallelogram they are half of just happens to be a *rectangular* parallelogram, but that is not so important. This means that moving from parallelograms to triangles (and not just right-angled ones) is now quite a small step. The hard part, of understanding about perpendicular height, has already been encountered in the easier case of parallelograms.

The problem with triangles like the one we began with in Figure 1 is that 'half the base times the height' doesn't seem to be very useful, because, even if we can use Pythagoras' Theorem to find the length of whichever side we choose as the base, the height seems tricky to find. That problem tends to push us towards the enclosing rectangle method outlined in Figure 2. But this isn't absolutely necessary. An alternative approach is to *shear* our triangle until it *becomes* an 'easy' triangle. For example, with the triangle shown in Figure 1, it takes only one shear to do this (see Figure 7). We have moved one of the vertices, parallel to the opposite side, so we haven't changed the base (the red side in Fig. 7) or the perpendicular height (because we were careful to move *parallel* to the red side). We have now created a triangle with one side parallel to the grid, so, taking this side now as the base, the area is $\frac{1}{2}(3 \times 2) = 3$. (A *GeoGebra* demonstration is at **www.geogebra.org/m/yvrd3jys**).

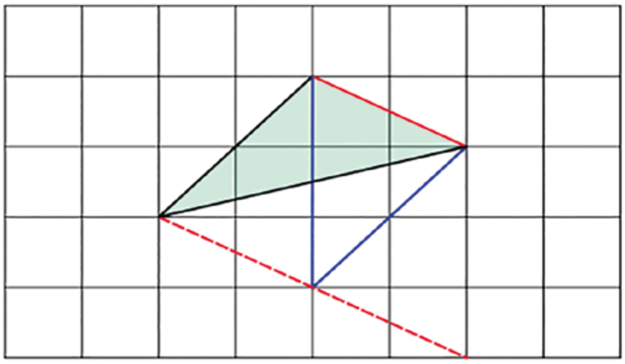

Figure 7. Shearing parallel to the red lines creates an 'easy' triangle.

Whether learners find this easier or not than the enclosing rectangle method will of course depend on how much shearing has been emphasised. (If this is the first time they meet shearing, then they will find it hard!) We like the shearing approach, not necessarily because it is 'quicker' or 'easier', but because it is highly visual, and the transformation leads to geometrical (not just numerical/ algebraic) simplicity. Looking for opportunities to *simplify* by *shearing* is a powerful strategy in many geometrical problems. For example, students sometimes work on tasks in which the perimeter of a shape is kept constant, but the area changes (e.g., a fixed length of fence enclosing different areas). With shearing, students can explore situations where the *area* remains constant but the *perimeter* of a shape changes.

There are many questions that students might explore. For example:

1. Can the area of this triangle (Fig. 1) be found by a *different* shear from the one shown in Figure 7? How many shearing solutions can you find?
2. Can the area of any triangle with its vertices on lattice points *always* be found by shearing? If so, is one shear always sufficient?
3. Can *any* parallelogram with it vertices on lattice points *always* be sheared into an easier one? Can it *always* be sheared into a rectangle with sides parallel to the grid? How many shears does it take?

Conclusion

We are not dismissing the value of the enclosing-rectangle method shown in Figure 1. This method really comes into its own with awkward-shaped polygons that have more than three sides or with tilted squares and other parallelograms. But we are moving in our thinking towards maybe seeing the enclosing-rectangle method as a method of last resort. To us, shearing seems more elegant and potentially a more conceptually-oriented approach that focuses attention helpfully on the important concept of a 'perpendicular height'.

Approaches to area based on shearing highlight the need for most geometrical situations to be visualised on a lattice, at least in the early stages. This avoids the problem of students being expected to make assumptions about the shapes they are presented with (e.g., that something that looks like a rectangle must be a rectangle, see Foster & Francome, 2022). When working *without* a grid, it is obvious that any non-right-angled triangle is always just one shear away from a right-angled triangle, which is a half-rectangle. When using a grid, this can be done in a precise, reasoned way, without assumptions or approximate measurement. And this does not need to be confined to square lattices; it works equally well on isometric or other shaped grids. These kinds of tasks reinforce the idea that area is measured in 'shaped-units', most commonly squared units, but that this is an arbitrary choice, convenient given a square lattice.

We think that using a shear is a more elegant way of finding areas, especially of triangles. It helps students see that area is the amount of surface enclosed within a boundary, rather than merely an answer to a calculation. We think it may help learners to acquire a more coherent view of area, rather than seeing it as a lot of disconnected formulae to remember.

Note

1. We are grateful to the Editors for pointing out that a horizontal line passing through the right-hand vertex of the triangle shown in Figure 1 divides it into two triangles with base 3 and height 1; hence total area 3.

References

Foster, C. 2019 'Trapezia acts', *Teach Secondary*, 8(4), pp. 82–83. **www.foster77.co.uk/Foster,%20Teach%20Secondary,%20Trapezia%20acts.pdf**

Foster, C., & Francome, T. 2022 'Diagrams not drawn accurately', *Mathematics in School*, 51(4), pp. 20–22.

Scott, P. 2006 'Area of lattice polygons', *The Australian Mathematics Teacher*, 62(3), pp. 2-5. **https://files.eric.ed.gov/fulltext/EJ922136.pdf**

Keywords: Area of triangle; Area of rectangle; Area of parallelogram; Area of polygon; Comparison of methods; Proof; Shearing.

Author: Chris Shore, Tom Francome and Colin Foster, all at Department of Mathematics Education, Schofield Building, Loughborough University, Loughborough LE11 3TU.

Email: **c.j.shore@lboro.ac.uk**
t.j.francome@lboro.ac.uk
c.foster@lboro.ac.uk

website: **www.foster77.co.uk**